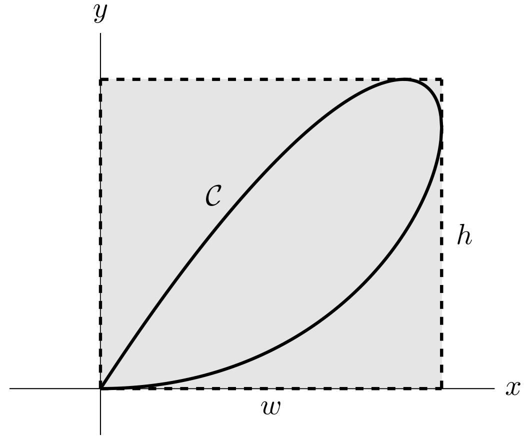
8. [8 points]

Let $\mathcal{C}$ be the curve implicitly defined by the equation

$$x^3 - 3xy + y^2 = 0.$$

The portion of the curve $\mathcal{C}$ that lies in the first quadrant is pictured to the right, not necessarily to scale, along with the smallest possible rectangle that contains it and has sides on the coordinate axes. This rectangle is shaded, and has side lengths w and h.



a. [4 points] Use implicit differentiation to find $\frac{dy}{dx}$.

Solution: Implicitly differentiating the given equation with respect to x, we get:

$$\begin{aligned} 3x^2 - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0, \\ \frac{dy}{dx}(2y - 3x) &= 3y - 3x^2, \\ \frac{dy}{dx} &= \frac{3(y - x^2)}{2y - 3x}. \end{aligned}$$

Answer: $\frac{dy}{dx} =$ $\frac{3(y - x^2)}{2y - 3x}$

b. [4 points] Find w and h, the width and height of the shaded rectangle containing $\mathcal{C}$. *Show all your work.*

Solution: The width w is the x-coordinate of the point on $\mathcal{C}$ with a vertical tangent line, and the height h is the y-coordinate of one of the points on $\mathcal{C}$ with a horizontal tangent line. Since $\frac{dy}{dx}$ is undefined where $\mathcal{C}$ has a vertical tangent line, and zero where $\mathcal{C}$ has a horizontal tangent line, we can find w and h, respectively, by setting the numerator and denominator of $\frac{dy}{dx}$ equal to zero and solving.

$$\frac{dy}{dx} = 0 \implies y = x^2 \implies x^3 - 3x(x^2) + (x^2)^2 = 0$$
$$\implies x^4 - 2x^3 = x^3(x-2) = 0.$$

This means $\mathcal{C}$ has a horizontal tangent line at $x = 2$ and $x = 0$, and from the picture we see that h is the y-coordinate of the point on $\mathcal{C}$ with x-coordinate 2. Plugging $x = 2$ into $y = x^2$, we get $h = y = 4$.

To find w, we find points on $\mathcal{C}$ where $\frac{dx}{dy} = 0$, or equivalently where $\frac{dy}{dx}$ is undefined. This happens when $y = \frac{3}{2}x$. Substituting $y = \frac{3}{2}x$ into the equation $x^3 - 3xy + y^2 = 0$, we get

$$0 = x^3 - 3x\left(\frac{3}{2}x\right) + \left(\frac{3}{2}x\right)^2 = x^3 - \frac{9}{2}x^2 + \frac{9}{4}x^2 = x^2\left(x - \frac{9}{4}\right).$$

Thus $\mathcal{C}$ has a vertical tangent line at $\left(\frac{9}{4}, \frac{27}{8}\right)$, so we get $w = \frac{9}{4}$ as the width of the rectangle.

Answer: $w =$ 9/4 and $h =$ 4