3. [9 points] A factory makes cylindrical cans of volume 400 cubic centimeters. Suppose the metal for the side of the can costs 1 cent per cm^2, and the metal for the top and the bottom costs 2 cents per cm^2. Find the **radius** of the can shape that minimizes the cost of producing such a can.

Show all your work, include units, and fully justify using calculus that you have in fact found the radius that minimizes cost.

Solution: Let r be the radius of the can, and h its height, both in centimeters. The volume of the can is then $\pi r^2 h$, so we obtain the constraint equation

$$400 = \pi r^2 h, \qquad \text{or} \qquad h = \frac{400}{\pi r^2} \tag{1}$$

after solving for h. Let C be the cost of producing a single can, in cents, so

$$C = 2\pi r h + 4\pi r^2 \tag{2}$$

since the cost of producing the side of a can is 1 cent times the surface area $2\pi r h$, while the cost of producing each of the top and bottom separately is 2 cents times the surface area πr^2.
We want to minimize C, so we use Equation (1) to rewrite our expression for C in terms of one variable. Substituting $\frac{400}{\pi r^2}$ for h in Equation (2) gives

$$C \;=\; C(r) \;=\; 2\pi r\left(\frac{400}{\pi r^2}\right) + 4\pi r^2 \;=\; \frac{800}{r} + 4\pi r^2. \tag{3}$$

The radius of the can must be positive, so we want to minimize the function $C = \frac{800}{\pi} + 4\pi r^2$ on the domain $(0, \infty)$. Differentiating with respect to r gives us

$$\frac{dC}{dr} \;=\; \frac{-800}{r^2} + 8\pi r. \tag{4}$$

Setting this derivative equal to 0 and solving for r, we get

$$\frac{-800}{r^2} + 8\pi r = 0, \qquad \text{so} \qquad 8\pi r^3 = 800, \qquad \text{thus} \qquad r = \left(\frac{100}{\pi}\right)^{1/3}.$$

So $r = \left(\frac{100}{\pi}\right)^{1/3}$ is the only critical point of $C(r)$. To verify that it is a global minimum of $C(h)$ on $(0, \infty)$, we note that

$$\lim_{x\to 0^+} C(r) \;=\; \lim_{x\to 0^+}\left(\frac{800}{r} + 4\pi r^2\right) \;=\; \infty,$$

and likewise

$$\lim_{x\to\infty} C(r) \;=\; \lim_{r\to\infty}\left(\frac{800}{r} + 4\pi r^2\right) \;=\; \infty.$$

Alternatively: one could check that $C'(r)$ is negative for $0 < r < \left(\frac{100}{\pi}\right)^{1/3}$ and positive for $\left(\frac{100}{\pi}\right)^{1/3} < r$ by plugging test points such as $r = 1$ and $r = 800$ into $C'(r)$; therefore, $\left(\frac{100}{\pi}\right)^{1/3}$ is a local minimum of $C(r)$ by the First Derivative Test, so it must in fact be a global minimum since it is the only critical point of $C(r)$ on $(0, \infty)$.
We conclude that $r = \left(\frac{100}{\pi}\right)^{1/3}$ cm is the radius that minimizes the cost of producing each can.

Answer: radius = $\left(\frac{100}{\pi}\right)^{1/3}$ **cm**