4. [10 points] Let $f(x)$ be the differentiable function defined by

$$f(x) = x^3 + \cos(x^3), \qquad \text{so} \qquad f'(x) = 3x^2\big(1 - \sin(x^3)\big).$$

For each part below, you must use calculus to find and justify your answers. Clearly state your conclusions and show enough evidence to support them. You may use the graphs of sine and cosine given on the front page, if necessary. Recall that $\pi \approx 3.14$.

a. [3 points] The function $f(x)$ has exactly three critical points **in the interval** $(-1, 2)$. Find them. Give exact answers, and *show your work.*

Solution: Critical points of $f(x)$ occur where $f'(x)$ is either zero or does not exist. Since $f(x)$ is differentiable everywhere, we just need to solve $f'(x) = 0$. From the given formula for $f'(x)$, we see that $f'(x) = 0$ when $x = 0$ or $\sin x^3 = 0$. We know the sine function is zero at

$$x \;=\; \ldots\; -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$$

but we know there are only three critical points in $(-1, 2)$ and $x = 0$ is one of them, so we must choose the two points above with cube roots in $(-1, 2)$. Since $-\frac{3\pi}{2} < -1$, also $\left(-\frac{3\pi}{2}\right)^{1/3} < -1$, so the other two critical points of $f(x)$ in $(-1, 2)$ must be $\left(\frac{\pi}{2}\right)^{1/3}$ and $\left(\frac{5\pi}{2}\right)^{1/3}$.

Answer: $f(x)$ has critical points at $x =$ **$0, \left(\frac{\pi}{2}\right)^{1/3}, \left(\frac{5\pi}{2}\right)^{1/3}$**

b. [4 points] Find the x-coordinates of all *local* minima and maxima of $f(x)$ **on the interval** $(-1, 2)$. If there are none of a particular type, write NONE. *Justify your answers.*

Solution: One solution is to notice that since $3x^2 \geq 0$ and $\sin(x^3) \leq 1$ for all x, the derivative of $f(x)$ is never negative, so $f(x)$ is an increasing function which means it has no local extrema.

Alternatively, we can apply the First Derivative Test to the critical points we found in part (a). Using sign logic and the fact that $3x^3 \geq 0$ and $1 - \sin(x^3) \geq 0$ for all x, we have

$f'(x)$: $+\cdot+=+$ | 0 | $+\cdot+=+$ | $\left(\frac{\pi}{2}\right)^{1/3}$ | $+\cdot+=+$ | $\left(\frac{5\pi}{2}\right)^{1/3}$ | $+\cdot+=+$

By the First Derivative Test, none of the critical points of $f(x)$ in the interval $(-1, 2)$ is a local extremum of $f(x)$.

Answer: Local min(s) at $x =$ **none** and Local max(es) at $x =$ **none**

c. [3 points] Find the x-coordinates of all *global* minima and maxima of $f(x)$ **on the interval** $[-1, 1]$. If there are none of a particular type, write NONE.

Solution: Since $f(x)$ is an increasing function, its global max on $[-1, 1]$ will occur at the right endpoint $x = 1$, while its global min on $[-1, 1]$ will occur at the left endpoint $x = -1$. Alternatively, we can evaluate $f(x)$ at the endpoints $x = \pm 1$ along with the sole critical point of $f(x)$ in $(-1, 1)$, namely $x = 0$:

$$f(-1) = -1 + \cos(-1), \qquad f(0) = 0 + 1 = 1, \qquad f(1) = 1 + \cos(1).$$

Since $\cos(-1) < 1$ and $\cos(1) > -1$, we see that $f(-1) < f(0) < f(1)$, so the global max of $f(x)$ on $[-1, 1]$ occurs at $x = 1$ and the global min at $x = -1$.

Answer: Global min(s) at $x =$ -1 and Global max(es) at $x =$ 1