7. [5 points] The equation $x^3+y^3-xy^2=5$ defines y implicitly as a function of x. Find a formula for $\dfrac{dy}{dx}$ in terms of x and y. Show every step of your work.

Solution: Implicitly differentiating both sides of $x^3+y^3-xy^2=5$ with respect to x gives

$$3x^2+3y^2\frac{dy}{dx}-y^2-2xy\frac{dy}{dx} = 0.$$

Rearranging terms and solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx}\Big(3y^2-2xy\Big) = y^2-3x^2, \qquad \text{so} \qquad \frac{dy}{dx} = \frac{y^2-3x^2}{3y^2-2xy}.$$

Answer: $\dfrac{dy}{dx} = \dfrac{y^2-3x^2}{3y^2-2xy}$

8. [8 points] Let $\mathcal{C}$ be the curve defined by the equation $x^2+y^3=8y$. Note that

$$\frac{dy}{dx}=\frac{2x}{8-3y^2}.$$

a. [4 points] Find the coordinates of all points (x,y) on the curve $\mathcal{C}$ where the tangent line to $\mathcal{C}$ is horizontal. Write your answer as a list of points in the form (x,y), or write NONE if there are no such points. *Show all your work.*

Solution: In order to find horizontal tangent lines, we set $\frac{dy}{dx}$ equal to zero and solve:

$$0 = \frac{dy}{dx} = \frac{2x}{8-3y^2} \qquad \text{when} \qquad x=0.$$

Now we must find all points on $\mathcal{C}$ such that $x=0$. So we substitute $x=0$ into the equation defining $\mathcal{C}$ to obtain $y^3=8y$. This equation has three solutions: $y=0$ and $y=\pm\sqrt{8}$.

Answer: $(0,0)$, $(0,\sqrt{8})$, $(0,-\sqrt{8})$

b. [4 points] The curve $\mathcal{C}$ intersects the line $y=1$ at exactly one point with a positive x value. Find an equation of the line tangent to the curve $\mathcal{C}$ at this point. *Show all your work.*

Solution: We plug $y=1$ into the equation that defines $\mathcal{C}$ to obtain $x^2+1=8$, or $x^2=7$. This has two solutions, $x=\pm\sqrt{7}$, and we want the positive one. So we are looking for an equation of the line tangent to $\mathcal{C}$ at the point $(\sqrt{7},1)$. We can find the slope of this line by plugging $x=\sqrt{7}$ and $y=1$ into the formula given for $\frac{dy}{dx}$ to obtain $\frac{dy}{dx}=\frac{2\sqrt{7}}{8-3(1)^2}=\frac{2\sqrt{7}}{5}$. Thus the tangent line has equation

$$L(x)=1+\frac{2\sqrt{7}}{5}\Big(x-\sqrt{7}\Big).$$

Answer: $y = 1+\frac{2\sqrt{7}}{5}(x-\sqrt{7})$