

7. [5 points] The equation $x^3 + y^3 - xy^2 = 5$ defines y implicitly as a function of x . Find a formula for $\frac{dy}{dx}$ in terms of x and y . Show every step of your work.

Solution: Implicitly differentiating both sides of $x^3 + y^3 - xy^2 = 5$ with respect to x gives

$$3x^2 + 3y^2 \frac{dy}{dx} - y^2 - 2xy \frac{dy}{dx} = 0.$$

Rearranging terms and solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} (3y^2 - 2xy) = y^2 - 3x^2, \quad \text{so} \quad \frac{dy}{dx} = \frac{y^2 - 3x^2}{3y^2 - 2xy}.$$

Answer: $\frac{dy}{dx} = \frac{y^2 - 3x^2}{3y^2 - 2xy}$

8. [8 points] Let $\mathcal{C}$ be the curve defined by the equation $x^2 + y^3 = 8y$. Note that

$$\frac{dy}{dx} = \frac{2x}{8 - 3y^2}.$$

- a. [4 points] Find the coordinates of all points (x, y) on the curve $\mathcal{C}$ where the tangent line to $\mathcal{C}$ is horizontal. Write your answer as a list of points in the form (x, y) , or write NONE if there are no such points. *Show all your work.*

Solution: In order to find horizontal tangent lines, we set $\frac{dy}{dx}$ equal to zero and solve:

$$0 = \frac{dy}{dx} = \frac{2x}{8 - 3y^2} \quad \text{when} \quad x = 0.$$

Now we must find all points on $\mathcal{C}$ such that $x = 0$. So we substitute $x = 0$ into the equation defining $\mathcal{C}$ to obtain $y^3 = 8y$. This equation has three solutions: $y = 0$ and $y = \pm\sqrt{8}$.

Answer: $(0, 0), (0, \sqrt{8}), (0, -\sqrt{8})$

- b. [4 points] The curve $\mathcal{C}$ intersects the line $y = 1$ at exactly one point with a positive x value. Find an equation of the line tangent to the curve $\mathcal{C}$ at this point. *Show all your work.*

Solution: We plug $y = 1$ into the equation that defines $\mathcal{C}$ to obtain $x^2 + 1 = 8$, or $x^2 = 7$. This has two solutions, $x = \pm\sqrt{7}$, and we want the positive one. So we are looking for an equation of the line tangent to $\mathcal{C}$ at the point $(\sqrt{7}, 1)$. We can find the slope of this line by plugging $x = \sqrt{7}$ and $y = 1$ into the formula given for $\frac{dy}{dx}$ to obtain $\frac{dy}{dx} = \frac{2\sqrt{7}}{8 - 3(1)^2} = \frac{2\sqrt{7}}{5}$. Thus the tangent line has equation

$$L(x) = 1 + \frac{2\sqrt{7}}{5}(x - \sqrt{7}).$$

Answer: $y = 1 + \frac{2\sqrt{7}}{5}(x - \sqrt{7})$