(10.) (11 points) You are designing a cylindrical bucket. The bucket must have a bottom, but it will have no lid, and you have 1000 square inches of steel sheet to use for the bucket.

If the radius of the bucket is r and the height is h, for what values of r and h does the bucket have maximum possible volume? What is this maximum volume? Show all your work, and clearly indicate your final answers below.

Let V be the volume of the bucket, and let S be the surface area of the bucket. We have that

$$V = \pi \cdot r^2 \cdot h, \quad S = \pi \cdot 2r \cdot h + \pi \cdot r^2.$$

Since we have 1000 square inches of sheet to work with, we have the equation

$$1000 = \pi \cdot 2r \cdot h + \pi \cdot r^2,$$

which we can solve for h:

$$h = \frac{1000 - \pi \cdot r^2}{\pi \cdot 2r}.$$

Now we can write V as a function of r only:

$$\begin{aligned} V(r) &= \pi \cdot r^2 \cdot \left(\frac{1000 - \pi \cdot r^2}{\pi \cdot 2r}\right) \\ &= 500r - \frac{\pi}{2} r^3 \end{aligned}$$

Now we optimize this function:

$$V'(r) = 500 - \frac{3\pi}{2} r^2 = 0,$$

so V has critical points at $r = \pm\sqrt{\frac{1000}{3\pi}}$. We restrict our attention to the positive critical point. Since V'' is negative at this critical point, we find that this positive value of r is a local maximum. Since we are only interested in values of r that are nonnegative, and since $V(0) = 0$, we conclude that $V(\sqrt{\frac{1000}{3\pi}})$ is the global maximum we seek. The final answers are obtained by plugging in $r = \sqrt{\frac{1000}{3\pi}}$:

$$h = \frac{1000 - \frac{1000}{3}}{\pi \cdot 2\sqrt{\frac{1000}{3\pi}}} = \frac{\frac{1000}{3}}{\sqrt{\frac{1000\pi}{3}}} = \frac{1000}{3} \cdot \frac{\sqrt{3}}{\sqrt{1000\pi}} = \sqrt{\frac{1000}{3\pi}}.$$

Thus, the volume is

$$V = \pi \left(\frac{1000}{3\pi}\right)^{\frac{3}{2}}$$

Optimal value of $r = \sqrt{\frac{1000}{3\pi}}$ inches

Optimal value of $h = \sqrt{\frac{1000}{3\pi}}$ inches

Maximum volume $= \pi \left(\frac{1000}{3\pi}\right)^{\frac{3}{2}}$ cubic inches