(8.) (10 points) Saruman the White is creating an army of orcs to cut down all the trees in Fangorn Forest. Saruman is currently trying to decide exactly how large the army should be in order to destroy the forest as quickly as possible.

The trouble is, orcs aren't very efficient. In very small armies they tend to work pretty well — one orc will emerge as the leader, and he will have good control over the others. They also organize fairly well in very large armies, once a military structure is established. In medium-sized armies, though, the orcs spend a lot of time fighting for dominance, and as a result they can't work very efficiently.

Saruman has noticed this, of course. His research indicates that an army of x *thousand* orcs, will be able to cut down

$$T(x) = \frac{x^3}{3} - 3x^2 + 8x \qquad \text{thousand trees per hour.}$$

(a) If Saruman is capable of producing an army of up to 3000 orcs, how many should he produce in order to maximize the hourly destruction of trees? [Saruman does not have a graphing calculator and must be convinced by the methods of calculus.]

We optimize $T(x)$:

$$\begin{aligned} T'(x) &= x^2 - 6x + 8 \\ &= (x-2)(x-4) \end{aligned}$$

So T has critical points at $x = 2$ and $x = 4$. We have that $T''(x) = 2x - 6$, and so we see that $x = 2$ is a local maximum, and $x = 4$ is a local minimum. We note that

$$T(2) = \frac{8}{3} - 12 + 16 = \frac{20}{3},$$

but we must also check endpoints: $T(0) = 0$, and $T(3) = 6$. We see that the maximum is at $x = 2$, so Saruman should produce an army of 2000 orcs.

(b) Does your answer change if Saruman can produce up to 4000 orcs? If so, how many should he produce now?

No, the answer does not change. $T(4)$ is a local minimum, so it must be less than $T(2)$. Since there are no other critical points between the old endpoint $x = 3$ and the new one, there is nothing else to check.

(c) Does your answer change if Saruman can produce up to 6000 orcs? If so, how many should he produce now?

The value of T at the new endpoint is

$$\begin{aligned} T(6) &= \frac{216}{3} - 108 + 48 \\ &= \frac{216}{3} - 60 \\ &= \frac{216}{3} - \frac{180}{3} \\ &= \frac{36}{3} \\ &= 12. \end{aligned}$$

This is larger than $T(2)$, so Saruman should now produce an army of 6000 orcs.