7. (4+2+3+6 points) You got a flight out, finally, but in your haste to leave, you locked Frosty the Snowman, Jr. in the warm greenhouse. Suppose $r(t)$ is the rate in cm^3/min that Frosty's volume is changing as he is trapped in the greenhouse. The time the doors of the greenhouse were closed corresponds to $t = 0$.

(a) Explain the meaning of the quantity $\int_2^5 r(t)dt$ in the context of this problem.

This is the total amount of volume Frosty has changed between 2 minutes after the door was closed up to 5 minutes. Since it only makes sense that Frosty is melting, we could say it is the amount of snow in cm^3 that Frosty melted between 2 and 5 minutes after the door was closed.

(b) What do you expect the sign of $r(t)$ to be for the meaningful domain of this problem? Why?

The sign should be negative indicating that Frosty is melting. It would not make sense for him to gain volume when he is in a warm greenhouse.

(c) If $r(t) = 3t^2 - 432$, what is the domain that makes sense for this problem? Why?

As explained in part (b) we want this to be negative. Therefore, we want $3t^2 - 432 < 0$. Solving this for t and noting that we would want a positive time, we need $0 \leq t \leq 12$ measured in minutes.

(d) Use the Fundamental Theorem of Calculus (and common sense) to determine the volume (in cm^3) of Frosty, Jr. when the door to the greenhouse was closed. Show all of your work and reasoning.

If we let $R(t)$ be the volume of Frosty, so that $R'(t) = r(t)$, then the Fundamental Theorem of Calculus gives us:

$$\int_0^{12} r(t)\, dt = R(12) - R(0).$$

We know that when $t = 12$ the domain ends, so we can assume that Frosty has completely melted at this point. (Don't worry, Santa can save him still!) Frosty's original volume is given by $R(0)$. So we have

$$\begin{aligned} R(0) &= -\int_0^{12} (3t^2 - 432)\, dt \\ &= -(t^3 - 432t)\,\big|_0^{12} \\ &= 3,456 \text{ cm}^3. \end{aligned}$$