6. (10 points) Using techniques from calculus, find the dimensions which will maximize the surface area of a solid circular cylinder whose height h and radius r, each in centimeters, are related by

$$h = 8 - \frac{r^2}{3}.$$

[Hint: the surface area of a cylinder is given by $2\pi r^2 + 2\pi rh$.]

Since the radius and height are related by $h = 8 - \frac{r^2}{3}$, the surface area may be written as a function of the radius, r, by

$$S(r) = 2\pi r^2 + 2\pi r(8 - \frac{r^2}{3}) = -\frac{2\pi}{3}r^3 + 2\pi r^2 + 16\pi r$$

To maximize the surface area we must find the global maximum of $S(r)$. We are only interested in positive values for r; i.e., the interval $r > 0$. We start by finding the critical points of $S(r)$. Note that $S'(r) = -2\pi r^2 + 4\pi r + 16\pi = -2\pi(r^2 - 2r - 8)$. To solve $S'(r) = 0$, we can factor. We get

$$-2\pi(r - 4)(r + 2) = 0$$

so the critical points occur at $r = 4$ and $r = -2$. On the interval $r > 0$, $r = 4$ is the only critical point. By the second derivative test, it is a local maximum ($S''(4) = -12\pi < 0$) Also, for very large values of r we see that $S(r)$ is negative so the global maximum occurs at $r = 4$. When $r = 4$ we find that $h = 8 - \frac{16}{3} = \frac{8}{3}$ and so these dimensions maximize the surface area.

$h = \frac{8}{3}$ cm ____________

$r = 4$ cm ____________