7. (14 points) Show your work!

(a) Confirm that

\[ F(x) = \frac{1}{4} x^4 \ln(x) - \frac{1}{16} x^4 + 12 \]

is an antiderivative for \( f(x) = x^3 \ln(x) \), for values of \( x > 0 \). Show your work.

We must show that \( F'(x) = f(x) \) for then \( F(x) \) is an antiderivative for \( f(x) \) by definition. To compute \( F'(x) \) we use the product rule:

\[ F'(x) = 4 \cdot \frac{1}{4} x^3 \ln(x) + \frac{1}{4} x^4 \cdot \frac{1}{x} - \frac{4}{16} x^3 = x^3 \ln(x) + \frac{1}{4} x^3 - \frac{1}{4} x^3 = x^3 \ln(x) = f(x). \]

(b) Use the Fundamental Theorem of Calculus to find \( \int_1^2 x^3 \ln(x) \, dx \). Give your answer in exact form—i.e., not a decimal approximation.

Since \( F(x) \) is an antiderivative for \( f(x) \) by part (a), the Fundamental Theorem states that

\[ \int_1^2 f(x) \, dx = F(2) - F(1). \]

By direct calculation,

\[ F(2) = \frac{1}{4} \cdot 16 \ln(2) - \frac{1}{16} 16 + 12 = 4 \ln(2) + 11 \]

while

\[ F(1) = \frac{1}{4} \ln(1) - \frac{1}{16} + 12 = 12 - \frac{1}{16}. \]

Therefore,

\[ \int_1^2 f(x) \, dx = 4 \ln(2) - \frac{15}{16} \]

(c) Find an equation of the tangent to the graph of \( F \) at \( x = 1 \).

Since \( F'(x) = f(x) \), the slope of the tangent line to \( F(x) \) at \( x = 1 \) is \( f(1) = 0 \). We know from part (b) that \( F(1) = 12 - \frac{1}{16} \), so the equation for the tangent line is

\[ y = 12 - \frac{1}{16} \]