7. (14 points) Show your work!

(a) Confirm that

$$F(x) = \frac{1}{4}x^4 \ln(x) - \frac{1}{16}x^4 + 12$$

is an antiderivative for $f(x) = x^3 \ln(x)$, for values of $x > 0$. Show your work.

We must show that $F'(x) = f(x)$ for then $F(x)$ is an antiderivative for $f(x)$ by definition. To compute $F'(x)$ we use the product rule:

$$F'(x) = 4\frac{1}{4}x^3 \ln(x) + \frac{1}{4}x^4\frac{1}{x} - \frac{4}{16}x^3 = x^3 \ln(x) + \frac{1}{4}x^3 - \frac{1}{4}x^3 = x^3 \ln(x) = f(x).$$

(b) Use the Fundamental Theorem of Calculus to find $\int_1^2 x^3 \ln(x)dx$. Give your answer in *exact form*– i.e., not a decimal approximation.

Since $F(x)$ is an antiderivative for $f(x)$ by part (a), the Fundamental Theorem states that

$$\int_1^2 f(x)dx = F(2) - F(1).$$

By direct calculation,

$$F(2) = \frac{1}{4}16\ln(2) - \frac{1}{16}16 + 12 = 4\ln(2) + 11$$

while

$$F(1) = \frac{1}{4}\ln(1) - \frac{1}{16} + 12 = 12 - \frac{1}{16}.$$

Therefore,

$$\int_1^2 f(x)dx = 4\ln(2) - \frac{15}{16}$$

(c) Find an equation of the tangent to the graph of F at $x = 1$.

Since $F'(x) = f(x)$, the slope of the tangent line to $F(x)$ at $x = 1$ is $f(1) = 0$. We know from part (b) that $F(1) = 12 - \frac{1}{16}$, so the equation for the tangent line is

$$y = 12 - \frac{1}{16}$$