3. (12 points) Consider the family of cubics of the form
\[ f(x) = ax^3 + bx + c \]
with \( a, b, \) and \( c \) non-zero constants.

(a) (2 points) Using the function \( f(x) = ax^3 + bx + c \) as given above, write the limit definition of the derivative function, \( f'(x) \). (No need to expand or simplify–just apply the definition to this function, using proper notation.)

\[
f'(x) = \lim_{h \to 0} \frac{a(x + h)^3 + b(x + h) + c - ax^3 - bx - c}{h}
\]

(b) (6 points) Under what conditions, if any, on \( a, b, \) and \( c \) will \( f \) have local extrema (i.e., maxima/minima)?

Since \( f \) is a polynomial, so critical points only occur when \( f'(x) = 0 \).

\[
f'(x) = 3ax^2 + b
\]

\( f'(x) = 0 \) if there are real values of \( x \) such that \( 2ax^2 + b = 0 \), this only occurs if \(-\frac{b}{2a} > 0 \) (or if \( a \) and \( b \) are opposite signs). If this conditions holds for \( a \) and \( b \) then there are two critical points at \( x = \pm \sqrt{-\frac{b}{2a}} \). To check that these are indeed extrema, one could take the second derivative \( f''(x) = 6ax \), and then \( f''(\pm \sqrt{-\frac{b}{2a}}) = \pm 6a \sqrt{-\frac{b}{3a}} \). Thus indeed, if \( a \) and \( b \) are opposite signs, then the second derivative is either positive or negative (since \( a \) and \( b \) are non-zero) at the critical points, and consequently each critical point is indeed an local extrema under this condition.

(c) (4 points) Under what conditions, if any, on \( a, b, \) and \( c \) will \( f \) have inflection point(s)?

Possible inflection points occur when \( f''(x) = 0 \). The second derivative of \( f(x) \) (as calculated above) is \( f''(x) = 6ax \). Since \( a \) is non-zero then the only possible inflection point occurs when \( x = 0 \). To check that this is indeed an inflection point we must check the concavity does indeed change at \( x = 0 \). This is indeed the case, for if \( a > 0 \) then \( f''(x) < 0 \) for \( x < 0 \) and \( f''(x) > 0 \) for \( x > 0 \). Consequently \( x = 0 \) is an inflection point for ANY values of \( a, b, \) and \( c \).