3. (12 points) Consider the family of cubics of the form

$$f(x) = ax^3 + bx + c$$

with a, b, and c non-zero constants.

(a) (2 points) Using the function $f(x) = ax^3 + bx + c$ as given above, write the **limit definition** of the derivative function, $f'(x)$. (No need to expand or simplify–just apply the definition to this function, using proper notation.)

$$f'(x) = \lim_{h \to 0} \frac{a(x+h)^3 + b(x+h) + c - ax^3 - bx - c}{h}$$

(b) (6 points) Under what conditions, if any, on a, b, and c will f have local extrema (i.e., maxima/minima)?

Since f is a polynomial, so critical points only occur when $f'(x) = 0$.

$$f'(x) = 3ax^2 + b$$

$f'(x) = 0$ if there are real values of x such that $2ax^2 + b = 0$, this only occurs if $-\frac{b}{3a} > 0$ (or if a and b are opposite signs). If this conditions holds for a and b then there are two critical points at $x = \pm\sqrt{-\frac{b}{3a}}$. To check that these are indeed extrema, one could take the second derivative $f''(x) = 6ax$, and then $f''(\pm\sqrt{-\frac{b}{3a}}) = \pm 6a\sqrt{-\frac{b}{3a}}$. Thus indeed, if a and b are opposite signs, then the second derivative is either positive or negative (since a and b are non-zero) at the critical points, and consequently each critical point is indeed an local extrema under this condition.

(c) (4 points) Under what conditions, if any, on a, b, and c will f have inflection point(s)?

Possible inflection points occur when $f''(x) = 0$. The second derivative of $f(x)$ (as calculated above) is $f''(x) = 6ax$. Since a is non-zero then the only *possible* inflection point occurs when $x = 0$. To check that this is indeed an inflection point we must check the concavity does indeed change at $x = 0$. This is indeed the case, for if $a > 0$ then $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$. Consequently $x = 0$ is an inflection point for ANY values of a, b, and c.