4. (10 points) Suppose a paraboloid cup is inscribed in a hemisphere of radius 4 inches. The volume of the paraboloid is given by \( \frac{1}{2} \pi r^2 h \). For what values of the parameters \( r \) and \( h \) is the volume of the cup maximized?

One can envision \( r \) and \( h \) being the coordinates of a point on a circle of radius 4, thus \( r \) and \( h \) must be related by:

\[
r^2 = 16 - h^2.
\]

Using this relationship and the given formula for the volume of the paraboloid, \( V = \frac{1}{2} \pi r^2 h \) we can write \( V \) in terms of \( h \) only by replacing \( r^2 \). Namely,

\[
V = \frac{1}{2} \pi (16 - h^2)h.
\]

Differentiating \( V \) with respect to \( h \) gives, \( dV/dh = \frac{1}{2} \pi (16 - 3h^2) \). Critical points occur at values of \( h \) when \( dV/dh = 0 \), such values of \( h \) are \( h = \pm \frac{4}{\sqrt{3}} \). Because \( h \) represents a height and since it must be less then 4 (since it is inscribed in the sphere), then we only are interested in \( 0 \leq h \leq 4 \). Clearly if \( h = 4 \) or \( h = 0 \) then the volume is 0. The only critical point between 0 and 4 is the positive critical point \( h = \frac{4}{\sqrt{3}} \), substituting back into the equation for the volume we see that when \( h = \frac{4}{\sqrt{3}} \), the volume is \( V = \frac{1}{2} \pi \left( 16 - \left( \frac{4}{\sqrt{3}} \right)^2 \right) \left( \frac{4}{\sqrt{3}} \right) = \frac{64\pi}{3\sqrt{3}} \). Consequently, since this volume is positive, it must be the maximum volume. We can find the radius at this value of \( h \) using \( r^2 = 16 - h^2 \) to get \( r = 4 \sqrt{\frac{2}{3}} \). Consequently, the volume is maximized when \( h = \frac{4}{\sqrt{3}} \) inches and \( r = 4 \sqrt{\frac{2}{3}} \) inches.