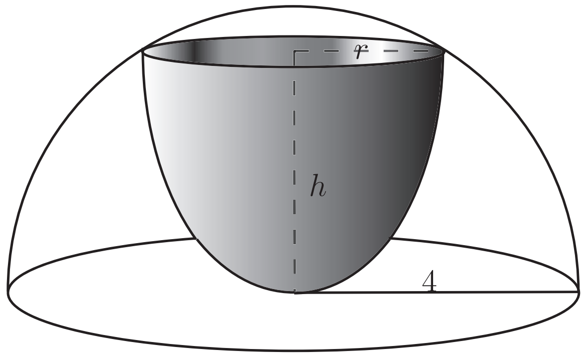
4. (10 points) Suppose a paraboloid cup is inscribed in a hemisphere of radius 4 inches. The volume of the paraboloid is given by $\frac{1}{2}\pi r^2 h$. For what values of the parameters r and h is the volume of the cup maximized?



One can envision r and h being the coordinates of a point on a circle of radius 4, thus r and h must be related by:

$$r^2 = 16 - h^2.$$

Using this relationship and the given formula for the volume of the paraboloid, V, ($V = \frac{1}{2}\pi r^2 h$) we can write V in terms of h only by replacing r^2. Namely,

$$V = \frac{1}{2}\pi(16 - h^2)h\,.$$

Differentiating V with respect to h gives, $dV/dh = \frac{1}{2}\pi(16-3h^2)$. Critical points occur at values of h when $dV/dh = 0$, such values of h are $h = \pm\frac{4}{\sqrt{3}}$. Because h represents a height and since it must be less then 4 (since it is inscribed in the sphere), then we only are interested in $0 \leq h \leq 4$. Clearly if $h = 4$ of $h = 0$ then the volume is 0. The only critical point between 0 and 4 is the positive critical point $h = \frac{4}{\sqrt{3}}$, substituting back into the equation for the volume we see that when $h = \frac{4}{\sqrt{3}}$, the volume is $V = \frac{1}{2}\pi\left(16 - \left(\frac{4}{\sqrt{3}}\right)^2\right)\left(\frac{4}{\sqrt{3}}\right) = \frac{64\pi}{3\sqrt{3}}$. Consequently, since this volume is positive, it must be the maximum volume. We can find the radius at this value of h using $r^2 = 16 - h^2$ to get $r = 4\sqrt{\frac{2}{3}}$. Consequently, the volume is maximized when $h = \frac{4}{\sqrt{3}}$ inches and $r = 4\sqrt{\frac{2}{3}}$ inches.