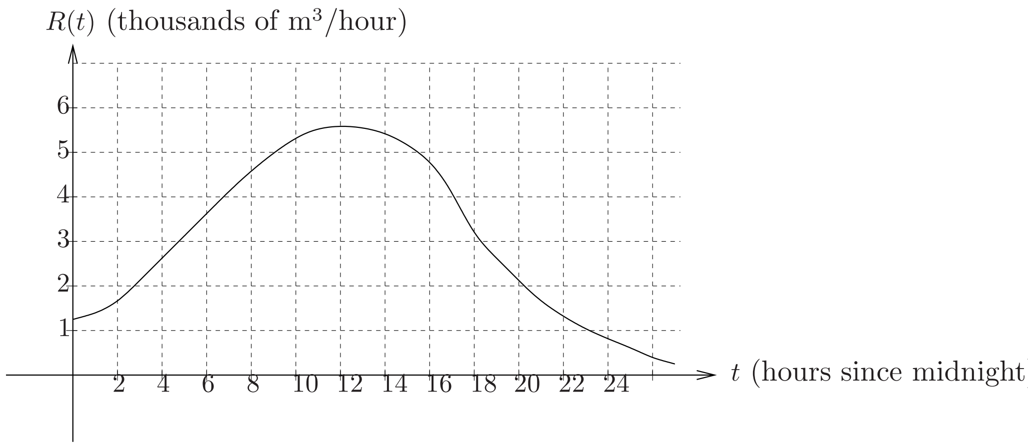
7. Marie is already tired of winter. She is dreaming of her grandparents' farm and days rafting on the river near the farm. Not all days on the river are beautiful, though. One summer a storm dumped about a year's worth of rainfall on the area in a couple of days. A man-made lake held back by a dam near the farm rose as the swollen rivers rushed toward the lake.

The graph below gives the rate R, in thousands of cubic meters per hour, that water was entering the lake during that day as a function of t, in hours since midnight. The volume of the lake at midnight was 400,000 cubic meters. The maximum volume that can be held by the dam is 460,000 cubic meters. Due to an oversight, the floodgates of the dam were kept closed until 6:00 a.m when they were opened to full capacity. The gates allowed water to leave the lake at a constant rate of 2000 cubic meters per hour.



(a) (4 points) Approximate the volume of the lake when the floodgates were opened. Show your reasoning.

Suppose $V(t)$ is the total volume at time t. Before the floodgates open $R(t) = V'(t)$. The volume at $t = 0$ is $400,000\text{m}^3$, thus $V(0) = 400$. The volume when the floodgates open is given by $V(6)$. By the fundamental theorem $V(6) = \int_0^6 R(t)dt + V(0) \approx 13 + 400 = 413$ thousands of m^3. The integral was approximated by counting approximately 6.5 squares (each of which has an area of 2) in the region below the graph of $R(t)$, above the t axis, and between $t = 0$ and $t = 6$.

(b) (4 points) When did the lake reach its highest volume? Explain

After the floodgates open the rate at which the volume is changing is given by $dV/dt =$(Rate in)-(Rate out)$=$ $(R(t) - 2)\frac{\text{thousands of m}^3}{\text{s}}$. Since at time $t = 6$ $R(t) > 2$ then $dV/dt > 0$ and thus the volume is increasing even though water is being let out. The volume continues to increase until $dV/dt = 0$ which occurs when $R(t) = 2$ (for $t > 6$). This happens at approximately $t = 20$ (or 8 pm). After that $R(t) < 2$ and $dV/dt < 0$ so the volume is decreasing. Thus the highest volume of the lake occurs at about 8 pm.

(c) (5 points) Approximately what was the highest volume of the lake on that day? Explain.

Since the highest volume occurs when $t = 20$ we can use the fundemental theorem of Calculus and the fact that $V'(t) = R(t)$ for $t < 6$ and $V'(t) = R(t) - 2$ for $6 \leq t \leq 20$, namely,

$$\begin{aligned} V(20) &= \int_0^{20} V'(t)dt + V(0) = \int_0^6 R(t)dt + \int_6^{20} (R(t) - 2)dt + V(0) \\ &= \int_0^{20} R(t)dt - 2(14) + 400 = 76 - 28 + 400 = 448 \text{ thousands of m}^3 \end{aligned}$$

Here the integral of $R(t)$ was approximated by counting approximately 38 squares (each of which has an area of 2) in the region below the graph of $R(t)$, above the t axis between $t = 0$ and $t = 20$.