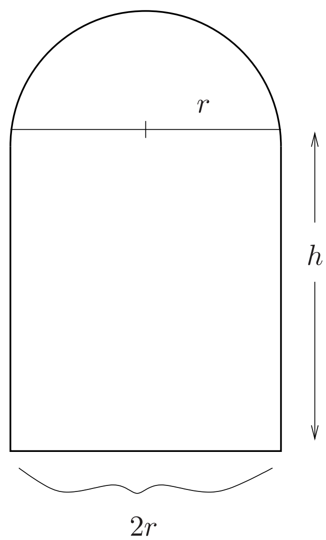
8. (14 points) Adrian is decorating her apartment for the holidays. She plans to outline a fake window with fancy lights. The window is to be in the shape of a rectangle topped by a semicircle (see Figure below). The cost of the fancy lights for the circular portion of the window is \$25 per meter, and the cost of the regular lights for the straight edging is \$10 per meter. If the "window" has a total area of 1 m^2, what are the dimensions that will minimize the cost of the lighted perimeter?



If r is the radius of the semicircle and h is the height of the straight side, then the area is

$$A = 2rh + \frac{\pi r^2}{2} = 1,$$

so

$$h = \frac{(1 - \frac{\pi r^2}{2})}{2r} = \frac{1}{2r} - \frac{\pi r}{4}.$$

Thus, the cost is

$$C = 25\pi r + 10\,(2r + 2h)\,,$$

which gives

$$C = 25\pi r + 10\left(2r + 2\left(\frac{1}{2r} - \frac{\pi r}{4}\right)\right) = 25\pi r + 20r + \frac{10}{r} - 5\pi r.$$

Then

$$C' = 25\pi + 20 - \frac{10}{r^2} - 5\pi.$$

$C'(r) = 0$ if $\frac{10}{r^2} = 20\pi + 20$, so

$$r^2 = \frac{1}{2\pi = 2} \approx 0.3475 \quad \text{meters.}$$

Checking for minimum, we have

$$C''(r) = \frac{10}{r^3}$$

which is greater than zero for all $r > 0$, so $r = 0.3475$ is a local minimum. Since this is the only critical point, it is the global minimum.

When $r = 0, 3475, h = 1.166$, so the dimensions are base $= 0.695$ meters, and height $= 1.166$ meters.