4. [11 points] Consider the function $f(x) = x^3 \ln x$.

a. [4 points] Use the general expression for a left-hand sum using 4 subdivisions to write an approximation for

$$\int_1^3 x^3 \ln x \, dx$$

—*i.e.*, express each term of the left-hand sum, using the given function. There is no need to evaluate the sum.

Solution: The width of each subdivision is given by $\Delta x = (b-a)/n = (3-1)/4 = 1/2$. We also know that the left hand sum, in this case, is given by $[f(a) + f(a+\Delta x) + f(a + 2\Delta x) + f(a+3\Delta x)]\Delta x$

$$= [1^3 \ln 1 + (1.5)^3 \ln 1.5 + 2^3 \ln 2 + 2.5^3 \ln 2.5](0.5)$$

.

b. [3 points] Show that $\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + C$. Show your work.

Solution: The statement $\int f(x)dx = F(x)$ is the same as the statement $F'(x) = f(x)$, so we need to take the derivative of the right hand side $F(x) = \frac{x^4}{4} \ln x - \frac{x^4}{16} + C$ and (hopefully) get the integrand $f(x) = x^3 \ln x$. We have that

$$F'(x) = x^3 \ln x + \frac{x^4}{4}\left(\frac{1}{x}\right) - \frac{x^3}{4} = x^3 \ln x + \frac{x^3}{4} - \frac{x^3}{4} = x^3 \ln x,$$

so, indeed, $\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + C$.

c. [4 points] Use the Fundamental Theorem of Calculus and part (b) to find the exact value of $\int_1^3 x^3 \ln x \, dx$. Leave your answer in *exact* form—in other words, do not convert to a decimal. Again, show your work.

Solution: By the Fundamental Theorem of Calculus, we know that $\int_a^b f(x)\, dx = F(b) - F(a)$, where $F'(x) = f(x)$. Applying this, and part (b), we have that $\int_1^3 x^3 \ln x \, dx = \left(\frac{x^4}{4} \ln x - \frac{x^4}{16}\right)\Big|_1^3$

$$= \frac{3^4}{16}(4 \ln 3 - 1) + \frac{1}{16}.$$