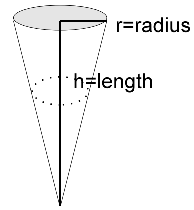
5. [13 points] A cone-shaped icicle is dripping from above the entrance to Dennison Hall. The icicle is melting at a rate of 1.2 cm^3 per hour. At 10:00 a.m., the icicle was 25 cm long and had a 2 cm radius at its widest point. Assume that the icicle keeps the same proportions as it melts. [Note: the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.]



a. [5 points] Determine the rate at which the length of the icicle is changing at 10:00 a.m.

Solution: We are given $\frac{dV}{dt} = -1.2$ and we want to find $\frac{dh}{dt}$ at 10:00 a.m. The proportions of the cone stay the same, so at any given time, $\frac{r}{h} = \frac{2}{25}$, so $r = \frac{2h}{25}$. Thus, $V = \frac{1}{3}\pi\left(\frac{2h}{25}\right)^2 \cdot h = \frac{1}{3}\pi\frac{4h^3}{625}$. Taking the derivative gives, $\frac{dV}{dt} = \frac{4\pi h^2}{625}\frac{dh}{dt}$, and at 10:00 a.m. we have

$$-1.2 = \frac{4\pi}{625}(25)^2\frac{dh}{dt} \text{ so that } \frac{dh}{dt} = \frac{-1.2}{4\pi} = -0.0955 \text{ cm/hour.}$$

b. [4 points] At what rate is the radius of the icicle changing at 10:00 a.m.?

Solution: From part (a), we have that $r = \frac{2h}{25}$, so $\frac{dr}{dt} = \frac{2}{25}\frac{dh}{dt}$. Thus, using $\frac{dh}{dt} = -0.0955$, we have $\frac{dr}{dt} = \frac{-0.0955(2)}{25} = -0.00764$ cm/hour.

c. [4 points] Let $V(t)$ and $r(t)$ denote the volume and radius, respectively, of the icicle t hours after 10:00 a.m. Assume that the icicle continued to melt from $t = 0$ (10:00 a.m.) to $t = M$. Circle all of the statements below that must be true if "After the icicle began dripping at 10:00 a.m., it took exactly M hours for the icicle to melt completely." [Circle the entire expression, and be certain that your circled answers are VERY clear!!]

i. $\int_0^M V'(t)\,dt > \int_0^{M/2} V'(t)\,dt$

ii. $\int_0^M V'(t)\,dt = 0$

iii. $\boxed{\int_0^M V'(t)\,dt = -V(0)}$

iv. $\int_0^2 r(t)\,dt = 0$

v. $\int_0^M r(t)\,dt = -2$

vi. $\boxed{\int_0^M r'(t)\,dt = -2}$

vii. $\int_2^0 V'(r)\,dr = M$

viii. $\int_0^M h(t)\,dt = 0$