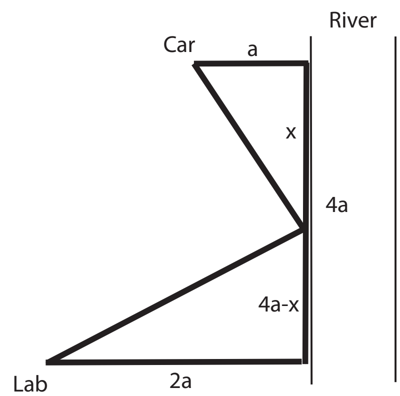
7. [12 points] After an unusual winter storm, the EPA is concerned about potential contamination of a river. A new researcher has been assigned the task of taking a sample to test the water quality. She tried to get as close to the river as possible in her car, but was forced to park a feet away. She also cannot get closer to the lab by car. She needs to walk to the river, retrieve a water sample, and then walk the sample to a lab located $4a$ feet down the river and $2a$ feet from the river bank.

If the researcher wants to walk as short a distance as possible, what path should she take as she walks from her car to the river and then from the river to the lab?

Solution: The shortest path will consist of a straight line from her car to the river and then from the river to the lab. Let P be the point on the river at which the researcher takes the sample of water, and let x be the number of miles downstream P is from where she parks. (See diagram below.)



The distance she walks is then given by the function $W(x) = \sqrt{a^2+x^2} + \sqrt{(2a)^2+(4a-x)^2}$. The goal is to minimize $W(x)$ on the interval $[0, 4a]$. Note that since W is continuous, the Extreme Value Theorem guarantees that W achieves a global maximum and global minimum on $[0, 4a]$.

$$W'(x) = \frac{x}{\sqrt{a^2+x^2}} - \frac{4a-x}{\sqrt{(2a)^2+(4a-x)^2}}$$

Since the denominators in this expression for $W'(x)$ are never 0 (for $a > 0$), critical points in $(a, 4a)$ occur only when $W'(x) = 0$. Solving this equation, we find
$\dfrac{x}{a^2+x^2} = \dfrac{4a-x}{\sqrt{(2a)^2+(4a-x)^2}}$ which reduces to $a^2(3x-4a)(x+4a) = 0$. Hence the only critical point of W on $(0, 4a)$ is $x = \dfrac{4a}{3}$.

Now, for $0 < x < \dfrac{4a}{3}$ (e.g. $x = a$), $W'(x) < 0$ and for $\dfrac{4a}{3} < x < 4a$ (e.g. $x = 2a$), $W'(x) > 0$. Hence W has a local minimum at $x = \dfrac{4a}{3}$ by the First Derivative Test. Since this is the unique critical point of W in the domain of interest, W achieves its global minimum on $[0, 4a]$ at $x = \dfrac{4a}{3}$.

So, in order to minimize the distance walked, the researcher should walk in a straight line from her car to the point on the river $\dfrac{4a}{3}$ feet downstream to retrieve a water sample and then from that point in a straight line to the lab.