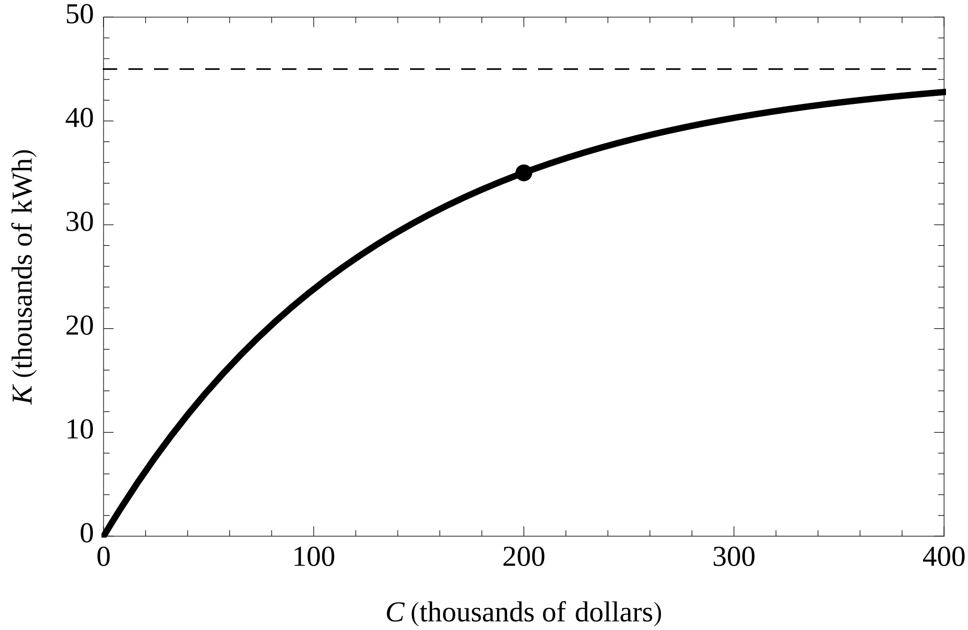
4. [14 points] Business owner Abbey Alexander is constructing a building for her latest business. Abbey wants her building to be energy efficient in order to save money on utility costs. Abbey has been given the following graph to help her decide on how much to spend on improvements.



In the graph, K is the expected savings in thousands of kilowatt hours (kWh) per year if Abbey spends C thousand dollars on energy-efficiency improvements. The dark point on the curve is (200, 35) and the dotted line is a horizontal asymptote at $K = 45$.

a. [5 points] Write a function of the form $K = a(1 - e^{-bC})$ for the curve in the graph above.

Solution: Since there is a horizontal asymptote at $K = 45$, we have

$$45 = \lim_{C\to\infty} K = a.$$

To find b we set

$$35 = 45(1 - e^{-200b}).$$

Solving for b, we have $b = \frac{\ln(9/2)}{200}$. Thus our equation for K is

$$K = 45(1 - e^{-\frac{\ln(9/2)}{200}C}).$$

b. [3 points] The current price of energy from Abbey's power company is \$250 per thousand kWh. Assuming this price stays constant, write a function $F(C)$ which gives Abbey's total savings on utility costs (in thousands of dollars) over the first 20 years.

Solution: Abbey's utility savings over the first 20 years will be

$$250 \cdot 20K = 250 \cdot 20 \cdot 45(1 - e^{-\frac{\ln(9/2)}{200}C}) = 225000(1 - e^{-\frac{\ln(9/2)}{200}C}).$$

The units here are dollars. If we want to write $F(C)$ in thousands of dollars (as required below) we must divide by 1000 giving $F(C) = 225(1 - e^{-\frac{\ln(9/2)}{200}C})$.

c. [6 points] If Abbey spends C thousand dollars on energy-efficiency improvements, her net monetary savings, N, over 20 years, is given by the formula

$$N = F(C) - C$$

where $F(C)$ is from part **b.** How much should Abbey spend on energy-efficiency improvements in order to maximize her net monetary savings over the first 20 years? Be sure to justify your answer.

Solution: We take the derivative of N seeking critical points.

$$N'(C) = 1.692e^{-\frac{\ln(9/2)}{200}C} - 1.$$

Solving for C, we have $C = 69.931$. Because

$$N''(C) = (-\frac{\ln(9/2)}{200})(1.692)(e^{-\frac{\ln(9/2)}{200}C}) = (-)(+)(+) < 0$$

for all $C \geq 0$, we know that N is concave down everywhere which means our solution, $C = 69.931$, is a global maximum. Thus Abbey should spend \$69,931 on improvements.