1. [14 points]
You are online playing the Facebook-based game, FarmVille, and you receive land with 5 stalks of corn on it. You decide that you would like to model the corn population on this patch of land using your calculus skills, so you recall that a good model for population growth is the logistic model

$$P(t) = \frac{L}{1 + Ae^{-kt}} \qquad L > 0, \quad A > 0, \quad k > 0.$$

a. [5 points] Using the ***limit definition of the derivative***, write an explicit expression for the derivative of the function $P(t)$ at $t = 1$. Do not evaluate this expression.

Solution:

$$P'(1) = \lim_{h \to 0} \frac{\frac{L}{1+Ae^{-k(1+h)}} - \frac{L}{1+Ae^{-k(1)}}}{h}$$

b. [5 points] Using the definition of the logistic model above, compute the following in terms of L, k, and A, showing your work or providing an explanation for each part:

i. [1 points] $\lim_{t \to \infty} P(t) = L$
Since the exponential piece is decreasing, it tends to 0 as $t \to \infty$. Then, the denominator goes to 1, and the limit of $P(t)$ as $t \to \infty$ is L.

ii. [1 points] $\lim_{t \to -\infty} P(t) = 0$
Since the exponential factor goes to infinity at $t \to -\infty$, the entire denominator goes to infinity, and the function tends to 0.

iii. [1 points] $P(0) = \frac{L}{1 + A}$
If we evaluate the function at 0, the exponential piece equals 1, so the denominator is just $1 + A$, giving the answer above.

iv. [2 points] $P'(0) = \frac{LAk}{(1 + A)^2}$
First, we see the derivative is $P'(t) = \frac{LAke^{-kt}}{(1 + Ae^{-kt})^2}$, so $P'(0) = \frac{LAk}{(1 + A)^2}$

c. [4 points] Your farmland satisfies the following conditions:

$$P(0) = 5, \ P'(0) = 1, \ \lim_{t \to \infty} P(t) = 100.$$

Based on your answers in part (b), compute the correct values of L, k, and A for the logistic equation modeling corn population on your land.

Solution: Since $\lim_{t\to\infty} P(t) = L$, the last piece of given information implies that $L = 100$. Since $P(0) = \dfrac{L}{1+A}$ and $L = 100$, we have from the given information

$$\frac{100}{1+A} = 5 \Rightarrow 20 = 1 + A \Rightarrow A = 19.$$

Finally, we use the last piece of information (combined with $L = 100$ and $A = 19$ to solve for k:

$$P'(0) = \frac{LAk}{(1+A)^2} = \frac{1900k}{400} = 1 \Rightarrow k = \frac{4}{19}$$

$L =$ 100 $A =$ 19 $k =$ $\frac{4}{19}$