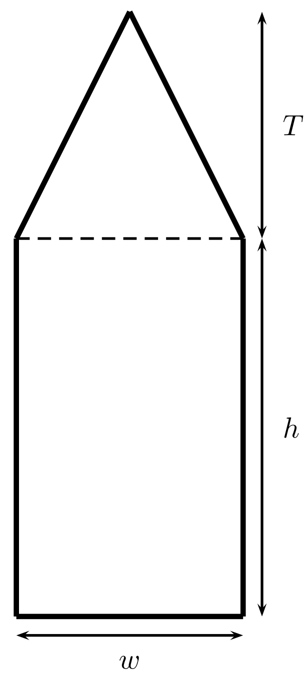
6. [10 points] Consider a window the shape of which is a rectangle of height h surmounted by a triangle having a height T that is two times the width w of the rectangle (see the figure below which is not drawn to scale). If the total area of the window is 5 square feet, determine the dimensions of the window which minimize the perimeter.



Solution: From the statement of the problem, we have $T = 2w$ and adding the areas of the triangle and the rectangle, we have the total area of the window to be

$$A = wh + 0.5Tw = wh + w^2 = 5.$$

Solving for h, this gives $h = 5/w - w$. To calculate the perimeter of the window, we must calculate the length, ℓ, of the two sides of the triangle which lay on the perimeter. by the Pythagorean theorem, we get $\ell = \sqrt{17}w/2$. Now the perimeter of the window is

$$P = 2h + w + 2\ell = \frac{10}{w} - 2w + w + \sqrt{17}w = \frac{10}{w} - w + \sqrt{17}w.$$

Setting the derivative of P equal to zero, we have

$$P' = \frac{-10}{w^2} - 1 + \sqrt{17}.$$

Which means $w = \sqrt{\frac{10}{-1+\sqrt{17}}}$ is our only critical point. The second derivative of P is $20w^{-3}$ which is positive for all $w > 0$. This means P is concave up everywhere. Since we only have one critical point, it must be a global minimum. Therefore the dimensions which minimize the perimeter of the window are $w = \sqrt{\frac{10}{-1+\sqrt{17}}}$ feet, $T = 2\sqrt{\frac{10}{-1+\sqrt{17}}}$ feet, and $h = 5\sqrt{\frac{-1+\sqrt{17}}{10}} - \sqrt{\frac{10}{-1+\sqrt{17}}}$ feet.