7. [14 points] For positive A and B, the force between two atoms is a function of the distance, r, between them:

$$f(r) = -\frac{A}{r^2} + \frac{B}{r^3} \qquad r > 0.$$

a. [2 points] Find the zeroes of f (in terms of A and B).

Solution: Finding a common denominator for f, we have

$$f(r) = \frac{-Ar + B}{r^3}.$$

This means $f(r) = 0$ when the numerator is zero, so $r = \frac{B}{A}$ is the only zero of f.

b. [7 points] Find the coordinates of the critical points and inflection points of f in terms of A and B.

Solution: Seeking critical points, we take the derivative of $f(r)$ and set it equal to zero

$$f'(r) = \frac{2A}{r^3} - \frac{3B}{r^4} = \frac{2Ar - 3B}{r^4} = 0.$$

Solving, we have that $r = \frac{3B}{2A}$ is our only critical point.
Now seeking inflection points, we take the second derivative of $f(r)$ and set it equal to zero.

$$f''(r) = -\frac{6A}{r^4} + \frac{12B}{r^5} = \frac{12B - 6Ar}{r^5} = 0.$$

Solving, we have that $r = \frac{2B}{A}$ is a candidate for an inflection point. Now we must test to see whether this is an inflection point. We compute $f''(\frac{B}{A}) = \frac{6B}{(B/A)^5} > 0$ since A and B are both positive, and also, $f''(\frac{3B}{A}) = \frac{-6B}{(3B/A)^5} < 0$ since A and B are both positive. This means f'' changes sign from positive to negative across the point $r = \frac{2B}{A}$, so it must be an inflection point.

c. [5 points] If f has a local minimum at $(1, -2)$ find the values of A and B. Using your values for A and B, justify that $(1, -2)$ is a local minimum.

Solution: We already know our only critical point is $r = \frac{3B}{2A}$. If f has a local minimum at $(1, -2)$, we must have that $1 = r = \frac{3B}{2A}$, so that $2A = 3B$. In addition, $-2 = f(1) = -A + B$. Solving these equations simultaneously, we have $A = 6$ and $B = 4$. We have already computed

$$f''(r) = \frac{12B - 6Ar}{r^5} = \frac{48 - 36r}{r^5}.$$

So $f''(1) = 48 - 36 = 12 > 0$ which means the critical point $(1, -2)$ is a local minimum since f is concave up at this point.