7. [14 points] For positive $A$ and $B$, the force between two atoms is a function of the distance, $r$, between them:

$$f(r) = -\frac{A}{r^2} + \frac{B}{r^3} \quad r > 0.$$ 

a. [2 points] Find the zeroes of $f$ (in terms of $A$ and $B$).

Solution: Finding a common denominator for $f$, we have

$$f(r) = -\frac{Ar}{r^3} + B.$$ 

This means $f(r) = 0$ when the numerator is zero, so $r = B^2/A$ is the only zero of $f$.

b. [7 points] Find the coordinates of the critical points and inflection points of $f$ in terms of $A$ and $B$.

Solution: Seeking critical points, we take the derivative of $f(r)$ and set it equal to zero

$$f'(r) = 2\frac{A}{r^3} - \frac{3B}{r^4} = \frac{2Ar - 3B}{r^4} = 0.$$ 

Solving, we have that $r = 3B^2/A$ is our only critical point.

Now seeking inflection points, we take the second derivative of $f(r)$ and set it equal to zero.

$$f''(r) = -\frac{6A}{r^4} + \frac{12B}{r^5} = \frac{12B - 6Ar}{r^5} = 0.$$ 

Solving, we have that $r = 2B/A$ is a candidate for an inflection point. Now we must test to see whether this is an inflection point. We compute $f''(2B/A) = \frac{6B}{(B/A)^5} > 0$ since $A$ and $B$ are both positive, and also, $f''(3B/A) = \frac{-6B}{(3B/A)^5} < 0$ since $A$ and $B$ are both positive. This means $f''$ changes sign from positive to negative across the point $r = 2B/A$, so it must be an inflection point.

c. [5 points] If $f$ has a local minimum at $(1, -2)$ find the values of $A$ and $B$. Using your values for $A$ and $B$, justify that $(1, -2)$ is a local minimum.

Solution: We already know our only critical point is $r = 3B^2/A$. If $f$ has a local minimum at $(1, -2)$, we must have that $1 = r = 3B^2/A$, so that $2A = 3B$. In addition, $-2 = f(1) = -A + B$. Solving these equations simultaneously, we have $A = 6$ and $B = 4$. We have already computed

$$f''(r) = \frac{12B - 6Ar}{r^5} = \frac{48 - 36r}{r^5}.$$ 

So $f''(1) = 48 - 36 = 12 > 0$ which means the critical point $(1, -2)$ is a local minimum since $f$ is concave up at this point.