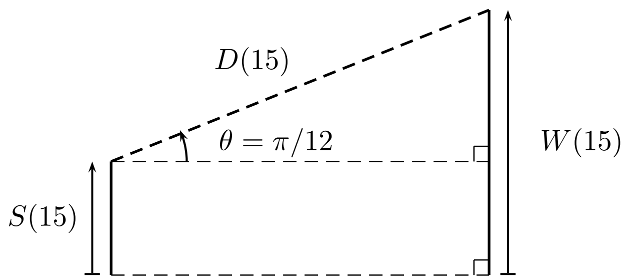
9. [10 points] You are sitting on a ship traveling at a constant speed of 6 ft/sec, and you spot a white object due east moving parallel to the ship. After tracking the object through your telescope for 15 seconds, you identify it as the white whale. Let $W(t)$ denote the distance of the whale from its starting point in feet, and $S(t)$ denote the distance of the ship from its starting point in feet, with t the time in seconds since you saw the whale. You also note that at the end of the 15 seconds you were tracking the whale, you have turned your head $\pi/12$ radians north to keep it in your sights.

a. [1 point] If initially the creature is 5280 ft (1 mile) from the ship due east, use the angle you have turned your head to find the distance $D(t)$ in feet between the ship and the creature at 15 seconds. The figure below should help you visualize the situation. Recall that, for right triangles, $\cos(\theta)$ is the ratio of the adjacent side to the hypotenuse.



Solution: Since $\cos(\pi/12) = \frac{5280}{D(15)}$, we find that $D(15) = \frac{5280}{\cos(\pi/12)} \approx 5466.258\text{ft}$.

b. [2 points] Let $\theta(t)$ give the angle you've turned your head after t seconds of tracking the whale. Write an equation $D(t)$ for the distance between the ship and the whale at time t (Hint: your answer may involve $\theta(t)$).

Solution: From the previous part, we know that the distance between the ship and the whale is 5280 divided by the cosine of the angle you've turned your head. Since $\theta(t)$ gives how far you've turned your head, we can find the distance at any time t using the function $D(t) = \frac{5280}{\cos(\theta(t))}$.

c. [3 points] If at precisely 15 seconds, you are turning your head at a rate of .01 radians per second, what is the instantaneous rate of change of the distance between the ship and the whale?

Solution: Since $D(t) = \frac{5280}{\cos(\theta(t))}$, we take the derivative with respect to t on both sides to get

$$\frac{dD}{dt} = \frac{5280\sin(\theta(t))}{\cos^2(\theta(t))}\theta'(t).$$

Since $\theta(15) = \pi/12$ and we are given $\theta'(15) = .01$, we get

$$\left.\frac{dD}{dt}\right|_{t=15} = \frac{5280\sin(\pi/12)}{\cos^2(\pi/12)}(.01) \approx 14.6468\text{ft/sec}.$$

d. [4 points] What is the speed of the whale at $t = 15$ seconds? Hint: Use the Pythagorean theorem.

Solution: The right triangle in the figure above has hypotenuse $D(t)$ and sides with length $D(t)$ and $W(t) - S(t)$, so the Pythagorean theorem states

$$D(t)^2 = 5280^2 + (W(t) - S(t))^2.$$

If we take the t derivative of both sides, we get

$$2D(t)\frac{dD}{dt} = 2(W(t) - S(t))\left(\frac{dW}{dt} - \frac{dS}{dt}\right).$$

To find $\left.\frac{dW}{dt}\right|_{t=15}$, we will need $D(15) = 5466.258$ and $\left.\frac{dD}{dt}\right|_{t=15} = 14.6468$. We also need to find $W(15) - S(15)$, but since this is one side of the right triangle, we can use tangent to find this distance: $W(15) - S(15) = 5280\tan(\pi/12) \approx 1414.7717$. Finally, we also need to know $\frac{dS}{dt}$, but in the description of the problem it says that ship is traveling at a constant speed of 6 ft/sec. Plugging all of this information into our equation we have

$$2(5466.258)(14.6468) = 2(1414.7717)\left(\frac{dW}{dt} - 6\right) \Rightarrow 56.5909 = \left.\frac{dW}{dt}\right|_{t=15} - 6.$$

Therefore, $\left.\frac{dW}{dt}\right|_{t=15} \approx 62.5909$ ft/sec