4. [14 points] Consider the family of functions

$$y = x^{2b} + ax^b$$

where a and b are nonzero constants.

a. [4 points] Calculate y'.

Solution: $y' = 2bx^{2b-1} + abx^{b-1}$

b. [4 points] Calculate y''.

Solution: $y'' = 2b(2b-1)x^{2b-2} + ab(b-1)x^{b-2}$

c. [6 points] Find values of a and b so that the resulting function has an inflection point at $(x, y) = (1, -4)$. Justify that $(1, -4)$ is an inflection point of the function with the values of a and b that you found.

Solution: The point $(1, -4)$ must be on the graph of our function, so

$$-4 = 1^{2b} + a \cdot 1^b = 1 + a.$$

Thus $a = -5$.
$(1, -4)$ must be an inflection point, so we plug $x = 1$ and $a = -5$ into our second derivative and set it equal to zero:

$$0 = 2b(2b-1) \cdot 1^{2b-2} - 5b(b-1) \cdot 1^{b-2} = 4b^2 - 2b - 5b^2 + 5b = -b^2 + 3b = -b(b-1).$$

So $b = 0$ or 3, but b is nonzero so $b = 3$.
To check that $(1, -4)$ is indeed and inflection point, we check that y'' changes sign at $x = 1$. Plugging in the values of a and b we found, we get

$$y'' = 6(5)x^4 - 5(3)(2)x^2 = 30x^4 - 30x^2 = 30x^2(x^2 - 1).$$

The term $30x^2$ is always positive and $x^2 - 1$ changes sign from negative to positive at $x = 1$. Therefore, $(1, -4)$ is an inflection point of the function.