7. [11 points] Consider the continuous function

$$f(x) = \begin{cases} x \cdot 2^{-x} & 1 \le x < 3, \\ \frac{1}{2-x} + \frac{11}{8} & 3 \le x \le 5. \end{cases}$$

Note that the domain of f is $[1, 5]$.

a. [7 points] Find the x-values of the critical points of f.

Solution: To find the critical points, we first take the derivatives of the two functions and set them equal to zero.

$$\frac{d}{dx} x2^{-x} = 2^{-x} - x \ln 2 \cdot 2^{-x} = 2^{-x}(1 - x \ln 2) = 0.$$

2^{-x} is never 0 so $1 = x \ln 2$ so $x = \frac{1}{\ln 2} \approx 1.44$, which is between 1 and 3, so it is a critical point.

$$\frac{d}{dx}\left(\frac{1}{2-x} + \frac{11}{8}\right) = \frac{1}{(2-x)^2},$$

which is never 0, but undefined at 2, which is not between 3 and 5 and therefore not a critical point.
To check if there is a critical point at $x = 3$, we can either graph the function on a calculator and see that there is a sharp corner there, or we can check and see that the derivatives of the two functions are not equal there:

$$2^{-3}(1 - 3 \ln 2) \approx -0.13 \neq 1 = \frac{1}{(2-3)^2}.$$

Thus, the critical points are at $x = \frac{1}{\ln 2}$ and $x = 3$.

b. [4 points] Find the y-values of the global maximum and global minimum of f if they exist, or explain why they don't exist.

Solution: Since this is a closed interval, we can just test the critical points and endpoints.
$f(1) = 1 \cdot 2^{-1} = 0.5$
$f\left(\frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \cdot 2^{-\frac{1}{\ln 2}} \approx 0.53$
$f(3) = 3 \cdot 2^{-3} = \frac{3}{8} = 0.375$
$f(5) = \frac{1}{2-5} + \frac{11}{8} = \frac{25}{24} \approx 1.042$
So the global maximum is $\frac{25}{24} \approx 1.042$ and the global minimum is $\frac{3}{8} = 0.375$.