7. [11 points] Consider the continuous function

\[ f(x) = \begin{cases} 
  x \cdot 2^{-x} & 1 \leq x < 3, \\
  \frac{1}{2^x} + \frac{11}{8} & 3 \leq x \leq 5.
\end{cases} \]

Note that the domain of \( f \) is \([1, 5] \).

a. [7 points] Find the \( x \)-values of the critical points of \( f \).

**Solution:** To find the critical points, we first take the derivatives of the two functions and set them equal to zero.

\[
\frac{d}{dx}x2^{-x} = 2^{-x} - x \ln 2 \cdot 2^{-x} = 2^{-x}(1 - x \ln 2) = 0.
\]

\( 2^{-x} \) is never 0 so \( 1 = x \ln 2 \) so \( x = \frac{1}{\ln 2} \approx 1.44 \), which is between 1 and 3, so it is a critical point.

\[
\frac{d}{dx} \left( \frac{1}{2^x} + \frac{11}{8} \right) = \frac{1}{(2 - x)^2}.
\]

which is never 0, but undefined at 2, which is not between 3 and 5 and therefore not a critical point.

To check if there is a critical point at \( x = 3 \), we can either graph the function on a calculator and see that there is a sharp corner there, or we can check and see that the derivatives of the two functions are not equal there:

\[
2^{-3}(1 - 3 \ln 2) \approx -0.13 \neq 1 = \frac{1}{(2 - 3)^2}.
\]

Thus, the critical points are at \( x = \frac{1}{\ln 2} \) and \( x = 3 \).

b. [4 points] Find the \( y \)-values of the global maximum and global minimum of \( f \) if they exist, or explain why they don’t exist.

**Solution:** Since this is a closed interval, we can just test the critical points and endpoints.

\[
\begin{align*}
 f(1) &= 1 \cdot 2^{-1} = 0.5 \\
 f \left( \frac{1}{\ln 2} \right) &= \frac{1}{\ln 2} \cdot 2^{-\frac{1}{\ln 2}} \approx 0.53 \\
 f(3) &= 3 \cdot 2^{-3} = \frac{3}{8} = 0.375 \\
 f(5) &= \frac{1}{2^5} + \frac{11}{8} = \frac{25}{24} \approx 1.042.
\end{align*}
\]

So the global maximum is \( \frac{25}{24} \approx 1.042 \) and the global minimum is \( \frac{3}{8} = 0.375 \).