9. [11 points] A cube of ice is removed from the freezer and begins to melt. Let \( \ell(t) \) be its side length, \( V(t) \) its volume, and \( S(t) \) its surface area, all dependent upon \( t \), the number of minutes since it was removed from the freezer. The ice cube is melting (its volume is changing) at a rate proportional to its surface area. That is, \( \frac{dV}{dt} = kS(t) \), for some number \( k \). Initially the ice cube has a side length of 2 inches.

a. [4 points] Write \( V \) and \( S \) in terms of \( \ell \). Calculate the rate of change (with respect to time) of the side length of the ice cube in terms of \( \ell \) and \( k \).

**Solution:** \( V = \ell^3 \) and \( S = 6\ell^2 \).

We know, by the chain rule, that

\[
\frac{dV}{dt} = \frac{dV}{d\ell} \frac{d\ell}{dt} = 3\ell^2 \frac{d\ell}{dt}.
\]

We are given that

\[
\frac{dV}{dt} = kS(t) = k6\ell^2.
\]

Setting these equal, we get

\[
3\ell^2 \frac{d\ell}{dt} = k6\ell^2.
\]

So \( \frac{d\ell}{dt} = 2k \) inches per minute.

b. [2 points] How fast is the volume of the ice cube changing immediately after it is removed from the freezer? Your answer will involve \( k \).

**Solution:** The side length at time \( t = 0 \) is 2 inches, so

\[
\left. \frac{dV}{dt} \right|_{t=0} = k \cdot 6 \cdot 2^2 = 24k \text{ in}^3 \text{ per minute}
\]

c. [2 points] What is the sign of \( k \)? Briefly explain.

**Solution:** From part (b) we see that \( k \) has the same sign as \( \left. \frac{dV}{dt} \right|_{t=0} \), which is negative, since the volume is decreasing. Thus, \( k \) must be negative as well.

d. [3 points] How long will it take the ice cube to melt completely? Your answer may involve \( k \).

**Solution:** The side length is changing at a constant rate of \( 2k \) and starts at 2 inches, so it will take \( T \) minutes, where \( 2 + 2kT = 0 \). So \( T = \frac{1}{k} \) minutes.