9. [11 points] A cube of ice is removed from the freezer and begins to melt. Let $\ell(t)$ be its side length, $V(t)$ its volume, and $S(t)$ its surface area, all dependent upon t, the number of minutes since it was removed from the freezer. The ice cube is melting (its volume is changing) at a rate proportional to its surface area. That is, $\frac{dV}{dt} = kS(t)$, for some number k. Initially the ice cube has a side length of 2 inches.

a. [4 points] Write V and S in terms of ℓ. Calculate the rate of change (with respect to time) of the side length of the ice cube in terms of ℓ and k.

Solution: $V = \ell^3$ and $S = 6\ell^2$.
We know, by the chain rule, that

$$\frac{dV}{dt} = \frac{dV}{d\ell}\frac{d\ell}{dt} = 3\ell^2\frac{d\ell}{dt}.$$

We are given that

$$\frac{dV}{dt} = kS(t) = k6\ell^2.$$

Setting these equal, we get

$$3\ell^2\frac{d\ell}{dt} = k6\ell^2.$$

So $\frac{d\ell}{dt} = 2k$ inches per minute.

b. [2 points] How fast is the **volume** of the ice cube changing immediately after it is removed from the freezer? Your answer will involve k.

Solution: The side length at time $t = 0$ is 2 inches, so
$\frac{dV}{dt}\big|_{t=0} = k \cdot 6 \cdot 2^2 = 24k$ in^3 per minute

c. [2 points] What is the sign of k? Briefly explain.

Solution: From part (b) we see that k has the same sign as $\frac{dV}{dt}\big|_{t=0}$, which is negative, since the volume is decreasing. Thus, k must be negative as well.

d. [3 points] How long will it take the ice cube to melt completely? Your answer may involve k.

Solution: The side length is changing at a constant rate of $2k$ and starts at 2 inches, so it will take T minutes, where $2 + 2kT = 0$. So $T = -\frac{1}{k}$ minutes.