5. [12 points] Consider the function

$$f(x) = (x - k)e^{-x/k}$$

where k is a positive constant. Note that the derivative of $f(x)$ is

$$f'(x) = e^{-x/k} - \frac{1}{k}(x - k)e^{-x/k}.$$

Your answers to this problem might involve the constant k.

Be sure to show all your work and justify all of your answers.

a. [7 points] Determine the global maximum and minimum values of $f(x)$ on the interval $[0, \infty)$. If $f(x)$ does not have a global maximum or a global minimum on this interval, explain why.

Solution: Begin by finding the critical points of $f(x)$. Since $f(x)$ is differentiable, we just need to find where $f'(x) = 0$.

$$f'(x) = e^{-x/k} - \frac{1}{k}(x - k)e^{-x/k} = e^{-x/k}\left(2 - \frac{x}{k}\right)$$

The factor $e^{-x/k}$ is always positive, and $2 - \frac{x}{k} = 0$ when $x = 2k$. So the only critial point is at $x = 2k$.

Test the critical point, endpoint, and end behavior:

$$f(2k) = ke^{-2}$$
$$f(0) = -k$$
$$\lim_{x\to\infty} f(x) = 0$$

So there is a global maximum value of ke^{-2} at $x = 2k$ and a global minimum value of $-k$ at $x = 0$.

b. [5 points] Find the x-coordinates of all inflection points of $f(x)$ on the domain $[0,\infty)$ or show that $f(x)$ does not have any inflection points on this interval.

Solution: The second derivative of f is

$$f''(x) = -\frac{1}{k}e^{-x/k} - \left(\frac{1}{k}e^{-x/k} - \frac{1}{k^2}(x-k)e^{-x/k}\right) = \frac{1}{k}e^{-x/k}\left(\frac{x}{k} - 3\right)$$

The factor $\frac{x}{k} - 3$ is equal to 0 when $x = 3k$. To show this is an inlection point, we can test $x = 2k$ and $x = 4k$:

$$f''(2k) = -\frac{1}{k}e^{-2} < 0$$
$$f''(4k) = \frac{1}{k}e^{-4} > 0$$

The second derivative changes sign at $x = 3k$ so this is an inflection point.