5. [12 points] Consider the function

\[ f(x) = (x - k)e^{-x/k} \]

where \( k \) is a positive constant. Note that the derivative of \( f(x) \) is

\[ f'(x) = e^{-x/k} - \frac{1}{k}(x - k)e^{-x/k}. \]

Your answers to this problem might involve the constant \( k \).

Be sure to show all your work and justify all of your answers.

a. [7 points] Determine the global maximum and minimum values of \( f(x) \) on the interval \([0, \infty)\). If \( f(x) \) does not have a global maximum or a global minimum on this interval, explain why.

Solution: Begin by finding the critical points of \( f(x) \). Since \( f(x) \) is differentiable, we just need to find where \( f'(x) = 0 \).

\[ f'(x) = e^{-x/k} - \frac{1}{k}(x - k)e^{-x/k} = e^{-x/k} \left( 2 - \frac{x}{k} \right) \]

The factor \( e^{-x/k} \) is always positive, and \( 2 - \frac{x}{k} = 0 \) when \( x = 2k \). So the only critical point is at \( x = 2k \).

Test the critical point, endpoint, and end behavior:

\[
\begin{align*}
  f(2k) &= ke^{-2} \\
  f(0) &= -k \\
  \lim_{x \to \infty} f(x) &= 0
\end{align*}
\]

So there is a global maximum value of \( ke^{-2} \) at \( x = 2k \) and a global minimum value of \( -k \) at \( x = 0 \).
b. [5 points] Find the $x$-coordinates of all inflection points of $f(x)$ on the domain $[0, \infty)$ or show that $f(x)$ does not have any inflection points on this interval.

**Solution:** The second derivative of $f$ is

$$f''(x) = -\frac{1}{k}e^{-x/k} - \left(\frac{1}{k}e^{-x/k} - \frac{1}{k^2}(x-k)e^{-x/k}\right) = \frac{1}{k}e^{-x/k}\left(\frac{x}{k} - 3\right)$$

The factor $\frac{x}{k} - 3$ is equal to 0 when $x = 3k$. To show this is an inflection point, we can test $x = 2k$ and $x = 4k$:

$$f''(2k) = -\frac{1}{k}e^{-2} < 0$$
$$f''(4k) = \frac{1}{k}e^{-4} > 0$$

The second derivative changes sign at $x = 3k$ so this is an inflection point.