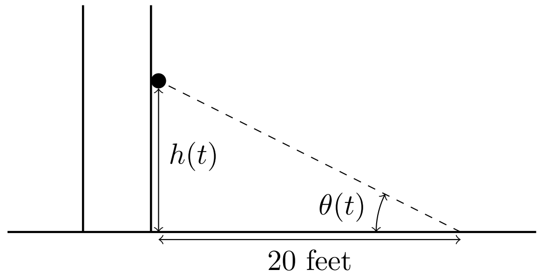
6. [12 points] Walking through Nichols Arboretum, you see a squirrel running down the trunk of a tree. The trunk of the tree is perfectly straight and makes a right angle with the ground. You stop 20 feet away from the tree and lie down on the ground to watch the squirrel. Suppose $h(t)$ is the distance in feet between the squirrel and the ground, and $\theta(t)$ is the angle in radians between the ground and your line of sight to the squirrel, with t being the amount of time in seconds since you stopped to watch the squirrel.



a. [3 points] Write an equation relating $h(t)$ and $\theta(t)$. (Hint: Use the tangent function.)

Solution:

$$\tan\theta(t) = \frac{h(t)}{20}$$

b. [5 points] If $\theta(t)$ is decreasing at 1/5 of a radian per second when $\theta(t) = \pi/3$, how fast is the squirrel moving at that time?

Solution: Differentiate with respect to t:

$$\frac{1}{\cos^2\theta(t)}\theta'(t) = \frac{h'(t)}{20}$$

Plug in $\theta'(t) = -1/5$ and $\theta(t) = \pi/3$:

$$\frac{1}{\cos^2(\pi/3)}(-1/5) = \frac{h'(t)}{20}$$

Solve to get $h'(t) = -16$ so the squirrel is moving at -16 feet per second.

c. [4 points] For the last second before the squirrel reaches the ground, it is moving at a constant speed of 20 feet per second. Suppose $\theta'(t) = -3/4$ at some point during this last second. How high is the squirrel at this time?

Solution: Start with

$$\frac{1}{\cos^2\theta(t)}\theta'(t) = \frac{h'(t)}{20}$$

and plug in $h'(t) = -20$ and $\theta'(t) = -3/4$:

$$\frac{1}{\cos^2\theta(t)}(-3/4) = \frac{-20}{20}$$

This gives us that $\cos^2\theta(t) = 3/4$. Then $\cos\theta(t) = \sqrt{3}/2$ (positive because $0 < \theta < \pi/2$), and $\theta(t) = \arccos(\sqrt{3}/2) = \pi/6$. Finally, we use the equation from part (a) to get

$$h(t) = 20\tan(\pi/6) = 20/\sqrt{3} \approx 11.547$$

so the squirrel is at a height of about 11.547 feet.