9. [8 points] Consider the family of functions given by
\[ I(t) = \frac{At^2}{B + t^2} \]
where \( A \) and \( B \) are positive constants. Note that the first and second derivatives of \( I(t) \) are
\[ I'(t) = \frac{2ABt}{(B + t^2)^2} \quad \text{and} \quad I''(t) = \frac{2AB(B - 3t^2)}{(B + t^2)^3}. \]

a. [2 points] Find \( \lim_{t \to \infty} I(t) \). Your answer may include the constants \( A \) and/or \( B \).

Answer: \( \lim_{t \to \infty} I(t) = \frac{A}{B} \)

A researcher studying the ice cover over Lake Michigan throughout the winter proposes that for appropriate values of \( A \) and \( B \), the function \( I(t) \) is a good approximation for the number of thousands of square miles of Lake Michigan covered by ice \( t \) days after the start of December. For such values of \( A \) and \( B \), a graph of \( y = I(t) \) for \( t \geq 0 \) is shown below.

![Graph of y = I(t)]

Based on observations, the researcher chooses values of the parameters \( A \) and \( B \) so that the following are true.

- \( y = 21 \) is a horizontal asymptote of the graph of \( y = I(t) \).
- \( I(t) \) is increasing the fastest when \( t = 25 \).

b. [6 points] Find the values of \( A \) and \( B \) for the researcher’s model.

Remember to show your work carefully.

**Solution:** From part (a) above, we know the graph of \( y = I(t) \) has a horizontal asymptote at \( y = A \). So \( A = 21 \).

\( I(t) \) is increasing fastest when \( I'(t) \) is maximized. For any value of \( t \) at which \( I'(t) \) is maximized, \( t \) is a critical point of \( I'(t) \), so \( I''(t) = 0 \) or \( I''(t) \) is undefined. The function \( I''(t) \) is defined for all \( t \), and \( I''(t) = 0 \) if and only if \( B - 3t^2 = 0 \). So since \( I'(t) \) is maximized when \( t = 25 \), we have \( B - 3(25)^2 = 0 \) so \( B = 3(25)^2 = 1875 \). (Alternatively, \( B - 3t^2 = 0 \) when \( t = \pm \sqrt{B/3} \). The positive solution is \( t = \sqrt{B/3} \), so \( \sqrt{B/3} = 25 \) and \( B = 1875 \).)

Thus, if \( A, B \) are chosen so that \( y = 21 \) is a horizontal asymptote of the graph, \( A = 21 \). If \( A, B \) are chosen so that \( I'(t) \) is maximized at \( t = 25 \), \( 25 = \sqrt{B/3} \). Thus, \( B = 1875 \).

Answer: \( A = \frac{21}{B} = 1875 \)