9. [8 points] Consider the family of functions given by

$$I(t) = \frac{At^2}{B + t^2}$$

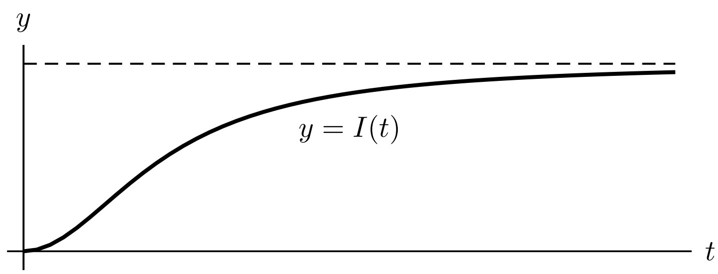
where A and B are positive constants. Note that the first and second derivatives of $I(t)$ are

$$I'(t) = \frac{2ABt}{(B+t^2)^2} \quad \text{and} \quad I''(t) = \frac{2AB(B-3t^2)}{(B+t^2)^3}.$$

a. [2 points] Find $\lim_{t\to\infty} I(t)$. *Your answer may include the constants A and/or B.*

Answer: $\lim_{t\to\infty} I(t) =$ A

A researcher studying the ice cover over Lake Michigan throughout the winter proposes that for appropriate values of A and B, the function $I(t)$ is a good approximation for the number of thousands of square miles of Lake Michigan covered by ice t days after the start of December. For such values of A and B, a graph of $y = I(t)$ for $t \geq 0$ is shown below.



Based on observations, the researcher chooses values of the parameters A and B so that the following are true.

- $y = 21$ is a horizontal asymptote of the graph of $y = I(t)$.
- $I(t)$ is increasing the fastest when $t = 25$.

b. [6 points] Find the values of A and B for the researcher's model.
Remember to show your work carefully.

Solution: From part (a) above, we know the graph of $y = I(t)$ has a horizontal asymptote at $y = A$. So $A = 21$.

$I(t)$ is increasing fastest when $I'(t)$ is maximized. For any value of t at which $I'(t)$ is maximized, t is a critical point of $I'(t)$, so $I''(t) = 0$ or $I''(t)$ is undefined. The function $I''(t)$ is defined for all t, and $I''(t) = 0$ if and only if $B - 3t^2 = 0$. So since $I'(t)$ is maximized when $t = 25$, we have $B - 3(25)^2 = 0$ so $B = 3(25)^2 = 1875$. (Alternatively, $B - 3t^2 = 0$ when $t = \pm\sqrt{B/3}$. The positive solution is t $t = \sqrt{B/3}$, so $\sqrt{B/3} = 25$ and $B = 1875$.)
Thus, if A, B are chosen so that $y = 21$ is a horizontal asymptote of the graph, $A = 21$. If A, B are chosen so that $I'(t)$ is maximized at $t = 25$, $25 = \sqrt{B/3}$. Thus, $B = 1875$.

Answer: $A =$ 21 and $B =$ 1875