10. [8 points] Gen is setting up a business selling hot chocolate in Srebmun Foyoj and, due to local restrictions, she will be able to produce and sell no more than 200 gallons. She has determined that the total cost, in dollars, for her to produce g gallons of hot chocolate can be modeled by

$$C(g) = \begin{cases} 100 + 90\sqrt{g} & \text{if } 0 \le g \le 100 \\ 400 - 10e^5 + 6g + 10e^{0.05g} & \text{if } 100 < g \le 200 \end{cases}$$

and that for $0 \le g \le 200$, the revenue, in dollars, that she will bring in from selling g gallons of hot chocolate is given by

$$R(g) = 15g.$$

a. [4 points] For what quantities of hot chocolate sold would Gen's marginal revenue equal her marginal cost?

Solution: We have $R'(g) = 15$ and $C'(g) = \begin{cases} 45g^{-1/2} & \text{if } 0 \le g < 100 \\ 6 + 0.5e^{0.05g} & \text{if } 100 < g \le 200. \end{cases}$

For $0 \le g < 100$, marginal revenue is therefore equal to marginal cost when $45g^{-1/2} = 15$, so $g^{1/2} = 3$ and $g = 9$. When $100 < g \le 200$,

$$\begin{aligned} 6 + 0.5e^{0.05g} &= 15 \\ e^{0.05g} &= 18 \\ 0.05g &= \ln(18) \\ g = 20\ln(18) &\approx 57.8 \end{aligned}$$

However, this value of g is not in the domain of this piece, so MC and MR are never equal on this piece.

Note: We can also conclude that no such point exists on this interval by noting that since $6 + 0.5e^{0.05 \cdot 100} > 80$ and MC is increasing, MC never equal 15.

Answer: 9 gallons

b. [4 points] Assuming Gen can sell up to 200 gallons of hot chocolate, how much hot chocolate should she produce in order to maximize her profit, and what would that maximum profit be? *You must use calculus to find and justify your answer. Be sure to provide enough evidence to justify your answer fully.*

Solution: Note that $C(g)$ is continuous, since $100 + 90\sqrt{100} = 1000$ and $400 - 10e^5 + 6 \cdot 100 + 10e^{0.05 \cdot 100} = 1000$.

The profit function is given by $\pi(g) = R(g) - C(g)$. Since both $R(g)$ and $C(g)$ are continuous, we may, by the Extreme Value Theorem, consider only critical points and endpoints of the domain. The endpoints are at $g = 0$ and $g = 200$, and the critical points are at $g = 9$ (by previous part) and $g = 100$ (where MR is undefined).

$$\begin{aligned} \pi(0) &= 0 - 100 = -100 \\ \pi(9) = 15 \cdot 9 - (100 + 90\sqrt{9}) &= 135 - 370 = -235 \\ \pi(100) &= 1500 - 1000 = 500 \\ \pi(200) &\approx -217{,}380 \end{aligned}$$

(The last is because $\pi(200) = 15 \cdot 200 - (400 - 10e^5 + 6 \cdot 200 + 10e^{0.05 \cdot 200})$
$= 3000 - (400 - 10e^5 + 1200 + 10e^{10}) \approx 3000 - 220380 = -217{,}380$.)

Therefore the max occurs at $g = 100$, which results in a profit of \$500.

Answer: gallons of hot chocolate: 100 and max profit: \$500