2. [10 points] Let a be a constant with $a > 1$.
A function $w(x)$ and its derivative $w'(x)$ are given below.

$$w(x) = a + \frac{x}{x^2 + a^2} \quad \text{and} \quad w'(x) = \frac{-(x-a)(x+a)}{(x^2+a^2)^2}.$$

a. [5 points] Find and classify the local extrema of $w(x)$. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank, write NONE if appropriate.

Solution: By inspection of the formula for w', we can see that the critical points of $w(x)$ are at $x = -a$ and $x = a$. (Note that $x^2 + a^2$ is never equal to 0 because $a \neq 0$.) The following sign chart shows the sign of $w'(x)$ on the intervals $-\infty < x < -a$, $-a < x < a$, and $a < x < \infty$. (Note that since a is positive, we have $-a < a$.)

Interval	$-\infty < x < -a$	$-a < x < a$	$a < x < \infty$
sign of $w'(x)$	$\frac{(-)(-)(-)}{+} = -$	$\frac{(-)(-)(+)}{+} = +$	$\frac{(-)(+)(+)}{+} = -$

By the First Derivative Test, we therefore see that $w(x)$ has a local minimum at $x = -a$ and a local maximum at $x = a$.

Answer: Local min(s) at $x =$ $-a$

Answer: Local max(es) at $x =$ a

b. [5 points] Find the global extrema of $w(x)$ on the interval $[1, \infty)$. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found the global extrema. For each answer blank, write NONE if appropriate.

Solution:
One solution: In part (a) we showed that $x = a$ is a local max. Since $x = a$ is the only critical point in the interval $[1, \infty)$, we conclude that $x = a$ is the global max on this interval. To determine the global min, we need to consider what happens at $x = 1$ and as $x \to \infty$. We have $\lim\limits_{x\to\infty} w(x) = a$. We also have $w(1) = a + \frac{1}{1+a^2}$, which is larger than a, so $x = 1$ is not the global min. Since $w(x)$ decreases to a (but never quite reaches it) as $x \to \infty$, we conclude that $w(x)$ has no global min on the interval $[1, \infty)$.

Another solution: We check the values of $w(x)$ at $x = 1$ and $x = a$ (note that since $a > 1$, a is always in the interval $[1, \infty)$, and $-a$ is never in this interval) and then consider what happens as x goes to infinity.
We have $w(1) = a + \frac{1}{1+a^2}$ and $w(a) = a + \frac{1}{2a}$. Thus we need to compare $\frac{1}{1+a^2}$ and $\frac{1}{2a}$. These are equal if $a = 1$ (but we know $a > 1$), and otherwise, $\frac{1}{2a}$ is larger. Since $w(x)$ is decreasing for $x > a$, the global max is $\frac{1}{2a}$.
We have $\lim\limits_{x\to\infty} w(x) = a$ but $w(x)$ is never equal to a. Since this limit is smaller than $w(1)$ and $w(a)$, we conclude that $w(x)$ has no global minimum on the interval $[1, \infty)$.

Answer: Global min(s) at $x =$ NONE

Answer: Global max(es) at $x =$ a