6. [5 points] Consider the differentiable function Z defined by

$$Z(v) = \begin{cases} \frac{e^{v-1} - v}{(v-1)^2} & \text{if } v \neq 1\\ \frac{1}{2} & \text{if } v = 1. \end{cases}$$

Use the limit definition of the derivative to write an explicit expression for Z'(1). Your answer should not involve the letter Z. Do not attempt to evaluate or simplify the limit. Please write your final answer in the answer box provided below.

Answer:
$$Z'(1) = \lim_{h \to 0} \frac{\frac{e^{1+h-1}-(1+h)}{(1+h-1)^2} - \frac{1}{2}}{h}$$
 or $\lim_{h \to 0} \frac{\frac{e^{h}-1-h}{h^2} - \frac{1}{2}}{h}$

7. [6 points] Consider the family of functions

$$g(x) = 16r^3\ln(|x|) + \frac{1}{3}k^3x^3$$

where r and k are nonzero constants. Note that

$$g'(x) = \frac{1}{x}(k^3x^3 + 16r^3)$$
 and $g''(x) = \frac{1}{x^2}(2k^3x^3 - 16r^3).$

Find values of r and k so that g(x) has an inflection point at (1,9). Be sure to justify that (1,9) is in fact an inflection point of g(x) for your choice of r and k.

Solution: The candidates for inflection points are the values of x in the domain of g(x) for which the second derivative is either zero or undefined. Since x = 0 is not in the domain of g(x), the only candidate is when $2k^3x^3 - 16r^3 = 0$, or when $x = \frac{2r}{k}$.

So, in order for g(x) to have an inflection point at x = 1, we must have $1 = \frac{2r}{k}$, or k = 2r. In order for the point (1,9) to lie on the graph of g(x), we need g(1) = 9. So we must have $g(1) = \frac{1}{3}k^3 = 9$, so k = 3, and $r = \frac{3}{2}$.

To justify that (1, 9) is really an inflection point of g(x), we will show that the second derivative changes sign across the point x = 1. If we plug in k = 3 and $r = \frac{3}{2}$ to g''(x), then we get

$$g''(x) = \frac{1}{x^2}(54x^3 - 54) = \frac{54}{x^2}(x^3 - 1).$$

When x > 1, $(x^3 - 1)$ is negative and $\frac{54}{x^2}$ is positive, so g''(x) is negative, and when x < 1, g''(x) is positive because all terms are positive. Thus, (1, 9) is indeed an inflection point.

Answer:
$$r = ________$$
 and $k = ______3$

Fall, 2015 Math 115 Exam 3 Problem 6 Solution