6. [5 points] Consider the differentiable function Z defined by

$$Z(v) = \begin{cases} \dfrac{e^{v-1} - v}{(v-1)^2} & \text{if } v \neq 1 \\ \frac{1}{2} & \text{if } v = 1. \end{cases}$$

Use the limit definition of the derivative to write an explicit expression for $Z'(1)$.
Your answer should not involve the letter Z. Do not attempt to evaluate or simplify the limit.
Please write your final answer in the answer box provided below.

Answer: $Z'(1) = \displaystyle\lim_{h\to 0} \frac{\frac{e^{1+h-1}-(1+h)}{(1+h-1)^2} - \frac{1}{2}}{h}$ or $\displaystyle\lim_{h\to 0} \frac{\frac{e^{h}-1-h}{h^2} - \frac{1}{2}}{h}$

7. [6 points] Consider the family of functions

$$g(x) = 16r^3 \ln(|x|) + \frac{1}{3}k^3x^3$$

where r and k are nonzero constants. Note that

$$g'(x) = \frac{1}{x}(k^3x^3 + 16r^3) \qquad \text{and} \qquad g''(x) = \frac{1}{x^2}(2k^3x^3 - 16r^3).$$

Find values of r and k so that $g(x)$ has an inflection point at $(1, 9)$. Be sure to justify that $(1, 9)$ is in fact an inflection point of $g(x)$ for your choice of r and k.

Solution: The candidates for inflection points are the values of x in the domain of $g(x)$ for which the second derivative is either zero or undefined. Since $x = 0$ is not in the domain of $g(x)$, the only candidate is when $2k^3x^3 - 16r^3 = 0$, or when $x = \frac{2r}{k}$.
So, in order for $g(x)$ to have an inflection point at $x = 1$, we must have $1 = \frac{2r}{k}$, or $k = 2r$.
In order for the point $(1, 9)$ to lie on the graph of $g(x)$, we need $g(1) = 9$. So we must have $g(1) = \frac{1}{3}k^3 = 9$, so $k = 3$, and $r = \frac{3}{2}$.

To justify that $(1, 9)$ is really an inflection point of $g(x)$, we will show that the second derivative changes sign across the point $x = 1$. If we plug in $k = 3$ and $r = \frac{3}{2}$ to $g''(x)$, then we get

$$g''(x) = \frac{1}{x^2}(54x^3 - 54) = \frac{54}{x^2}(x^3 - 1).$$

When $x > 1$, $(x^3 - 1)$ is negative and $\frac{54}{x^2}$ is positive, so $g''(x)$ is negative, and when $x < 1$, $g''(x)$ is positive because all terms are positive. Thus, $(1, 9)$ is indeed an inflection point.

Answer: $r = \frac{3}{2}$ and $k = 3$