2. [9 points] Uri is filling a cone with molten aluminum. The cone is upside-down, so the “base” is at the top of the cone and the vertex at the bottom, as shown in the diagram. The base is a circular disk with radius 7 cm and the height of the cone is 12 cm. Recall that the volume of a cone is \( \frac{1}{3}Ah \), where \( A \) is the area of the base and \( h \) is the height of the cone (i.e., the vertical distance from the vertex to the base). (Note that the diagram may not be to scale.)

a. [3 points] Write a formula in terms of \( h \) for the volume \( V \) of molten aluminum, in cm\(^3\), in the cone if the molten aluminum in the cone reaches a height of \( h \) cm. 

**Solution:** Let \( r \) be the radius of the top surface of the molten aluminum. Using similar triangles, we see \( r = \frac{7h}{12} \). Since the top surface of the molten aluminum is a circular disk, its area is \( \pi r^2 \). So \( V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left( \frac{7}{12} \right)^2 h^3 \left( = \frac{49\pi}{432} h^3 \right) \).

**Answer:** \( V = \frac{\pi}{3} \left( \frac{7}{12} \right)^2 h^3 \)

b. [3 points] The height of molten aluminum is rising at 3 cm/sec at the moment when the molten aluminum in the cone has reached a height of 11 cm. What is the rate, in cm\(^3\)/sec, at which Uri is pouring molten aluminum into the cone at that moment?

**Solution:** We differentiate \( V = \frac{\pi}{3} \left( \frac{7}{12} \right)^2 h^3 \) with respect to \( h \) to get \( \frac{dV}{dh} = \pi \left( \frac{7}{12} \right)^2 h^2 \). So, \( \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = \pi \left( \frac{7}{12} \right)^2 h^2 \cdot \frac{dh}{dt} \).

(Alternatively, differentiating both sides of the equation \( V = \frac{\pi}{3} \left( \frac{7}{12} \right)^2 h^3 \) with respect to \( t \) results in the same formula for \( \frac{dV}{dt} \).)

We are given that \( \frac{dh}{dt} \bigg|_{h=11} = 3 \), so we find 

\[
\frac{dV}{dt} \bigg|_{h=11} = \pi \left( \frac{7}{12} \right)^2 11^2 \cdot 3 = \frac{17787\pi}{144} = \frac{5929\pi}{48} \approx 388.052.
\]

Thus at the moment when the molten aluminum in the cone has reached a height of 11 cm, Uri is pouring molten aluminum into the cone at a rate of \( \frac{5929\pi}{48} \) (or about 388.052) cm\(^3\)/sec

**Answer:** \( \pi \left( \frac{7}{12} \right)^2 11^2 \cdot 3 = \frac{5929\pi}{48} \approx 388.052 \)

c. [3 points] The height of molten aluminum is rising at 3 cm/sec at the moment when the molten aluminum in the cone has reached a height of 11 cm. What is the rate, in cm\(^2\)/sec, at which the area of the top surface of the molten aluminum is increasing at that moment?
Solution: Let $A$ be the area of the top surface of the molten aluminum. Since $A = \pi r^2 = \pi \left(\frac{7}{12}\right)^2 h^2$, we see that $\frac{dA}{dh} = 2\pi \left(\frac{7}{12}\right)^2 h$.

So, $\frac{dA}{dt} = \frac{dA}{dh} \frac{dh}{dt} = 2\pi \left(\frac{7}{12}\right)^2 h \cdot \frac{dh}{dt}$.

(Alternatively, differentiating both sides of the equation $A = \pi \left(\frac{7}{12}\right)^2 h^2$ with respect to $t$ results in the same formula for $\frac{dA}{dt}$.)

We are given that $\frac{dh}{dt} \bigg|_{h=11} = 3$, so we find

$\frac{dA}{dt} \bigg|_{h=11} = 2\pi \left(\frac{7}{12}\right)^2 11 \cdot 3 = \frac{3234\pi}{144} = \frac{539\pi}{24} \approx 70.5549$.

Thus at the moment when the molten aluminum in the cone has reached a height of 11 cm, the area of the top surface of the molten aluminum is increasing at a rate of $\frac{539\pi}{24}$ (or about 70.5549) cm$^2$/sec.

Answer: $\frac{22\pi \left(\frac{7}{12}\right)^2 \cdot 3}{24} = \frac{539\pi}{24} \approx 70.5549$