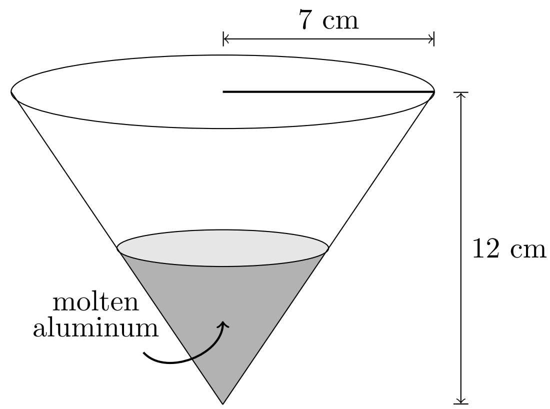
2. [9 points] Uri is filling a cone with molten aluminum. The cone is upside-down, so the "base" is at the top of the cone and the vertex at the bottom, as shown in the diagram. The base is a circular disk with radius 7 cm and the height of the cone is 12 cm.
Recall that the volume of a cone is $\frac{1}{3}Ah$, where A is the area of the base and h is the height of the cone (i.e., the vertical distance from the vertex to the base). (Note that the diagram may not be to scale.)



a. [3 points] Write a formula in terms of h for the volume V of molten aluminum, in cm^3, in the cone if the molten aluminum in the cone reaches a height of h cm.

Solution: Let r be the radius of the top surface of the molten aluminum. Using similar triangles, we see $r = \dfrac{7h}{12}$. Since the top surface of the molten aluminum is a circular disk, its area is πr^2.
So $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\left(\dfrac{7}{12}\right)^2 h^3 \left(= \dfrac{49\pi}{432}h^3\right)$.

Answer: $V =$ $\dfrac{\pi}{3}\left(\dfrac{7}{12}\right)^2 h^3$

b. [3 points] The height of molten aluminum is rising at 3 cm/sec at the moment when the molten aluminum in the cone has reached a height of 11 cm. What is the rate, in cm^3/sec, at which Uri is pouring molten aluminum into the cone at that moment?

Solution: We differentiate $V = \dfrac{\pi}{3}\left(\dfrac{7}{12}\right)^2 h^3$ with respect to h to get $\dfrac{dV}{dh} = \pi\left(\dfrac{7}{12}\right)^2 h^2$.
So, $\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} = \pi\left(\dfrac{7}{12}\right)^2 h^2 \cdot \dfrac{dh}{dt}$.
(Alternatively, differentiating both sides of the equation $V = \dfrac{\pi}{3}\left(\dfrac{7}{12}\right)^2 h^3$ with respect to t results in the same formula for $\dfrac{dV}{dt}$.)
We are given that $\left.\dfrac{dh}{dt}\right|_{h=11} = 3$, so we find
$\left.\dfrac{dV}{dt}\right|_{h=11} = \pi\left(\dfrac{7}{12}\right)^2 11^2 \cdot 3 = \dfrac{17787\pi}{144} = \dfrac{5929\pi}{48} \approx 388.052.$
Thus at the moment when the molten aluminum in the cone has reached a height of 11 cm, Uri is pouring molten aluminum into the cone at a rate of $\dfrac{5929\pi}{48}$ (or about 388.052) cm^3/sec

Answer: $\pi\left(\dfrac{7}{12}\right)^2 11^2 \cdot 3 = \dfrac{5929\pi}{48} \approx 388.052$

c. [3 points] The height of molten aluminum is rising at 3 cm/sec at the moment when the molten aluminum in the cone has reached a height of 11 cm. What is the rate, in cm^2/sec, at which the area of the top surface of the molten aluminum is increasing at that moment?

Solution: Let A be the area of the top surface of the molten aluminum.
Since $A = \pi r^2 = \pi \left(\frac{7}{12}\right)^2 h^2$, we see that $\frac{dA}{dh} = 2\pi \left(\frac{7}{12}\right)^2 h$.
So, $\frac{dA}{dt} = \frac{dA}{dh}\frac{dh}{dt} = 2\pi \left(\frac{7}{12}\right)^2 h \cdot \frac{dh}{dt}$.
(Alternatively, differentiating both sides of the equation $A = \pi \left(\frac{7}{12}\right)^2 h^2$ with respect to t results in the same formula for $\frac{dA}{dt}$.)
We are given that $\left.\frac{dh}{dt}\right|_{h=11} = 3$, so we find
$\left.\frac{dA}{dt}\right|_{h=11} = 2\pi \left(\frac{7}{12}\right)^2 11 \cdot 3 = \frac{3234\pi}{144} = \frac{539\pi}{24} \approx 70.5549$.
Thus at the moment when the molten aluminum in the cone has reached a height of 11 cm, the area of the top surface of the molten aluminum is increasing at a rate of $\frac{539\pi}{24}$ (or about 70.5549) cm^2/sec.

Answer: $22\pi \left(\frac{7}{12}\right)^2 \cdot 3 = \frac{539\pi}{24} \approx 70.5549$