10. [8 points] Consider the family of functions \( g(x) = e^x - kx \), where \( k \) is a positive constant.

a. [2 points] Show that the point \( \left( \ln(k), k - k\ln(k) \right) \) is the only critical point of \( g(x) \) for all positive \( k \). Show all your work to receive full credit.

Solution: \( g'(x) = e^x - k \), then \( g'(x) = 0 \) if \( e^x = k \) or \( x = \ln(k) \). There are no other critical points since \( g'(x) \) is defined for all values of \( x \).

The \( y \)-coordinate of the critical points is given by \( g(\ln(k)) = e^{\ln(k)} - k\ln(k) = k - k\ln(k) \).

b. [2 points] Show that \( g(x) \) has a global minimum on \( (-\infty, \infty) \) at \( x = \ln(k) \). Use calculus to justify your answer.

Solution: Since \( e^x - kx \to \infty \) as \( x \to \infty \) and \( e^x - kx \to \infty \) as \( x \to -\infty \) and \( x = \ln(k) \) is the only critical point, then it is a global minimum.

c. [4 points] Find all values of \( 0.5 \leq k \leq 2 \) that maximize the \( y \)-value of the global minimum of \( g(x) \) on \( (-\infty, \infty) \). Use calculus to justify your answer. Write NONE if no such value exists.

Solution: Let \( h(k) = k - k\ln(k) \) defined on \( 0.5 \leq k \leq 2 \). To find critical points, we find where \( h'(k) = -\ln(k) = 0 \). This occurs at \( k = 1 \). Looking at the table of values

\[
\begin{array}{c|c|c}
  k & 0.5 & 2 \\
  \hline
  h(k) & 0.5(1 - \ln(0.5)) \approx 0.84 & 1(2 - \ln(2)) \approx 0.61 \\
\end{array}
\]

Then the value of \( k \) that maximizes the value of the global minimum of \( g(x) \) is \( k = 1 \).

Answer: \( k = 1 \).