10. [8 points] Consider the family of functions $g(x) = e^x - kx$, where k is a positive constant.

a. [2 points] Show that the point $\big(\ln(k)\ ,\ k - k\ln(k)\big)$ is the only critical point of $g(x)$ for all positive k. Show all your work to receive full credit.

Solution: $g'(x) = e^x - k$, then $g'(x) = 0$ if $e^x = k$ or $x = \ln(k)$. There are no other critical points since $g'(x)$ is defined for all values of x.
The y-coordinate of the critical points is given by $g(\ln(k)) = e^{\ln(k)} - k\ln(k) = k - k\ln(k)$.

b. [2 points] Show that $g(x)$ has a global minimum on $(-\infty, \infty)$ at $x = \ln(k)$. Use calculus to justify your answer.

Solution: Since $e^x - kx \to \infty$ as $x \to \infty$ and $e^x - kx \to \infty$ as $x \to -\infty$ and $x = \ln(k)$ is the only critical point, then it is a global minimum.

c. [4 points] Find all values of $0.5 \le k \le 2$ that maximize the y-value of the global minimum of $g(x)$ on $(-\infty, \infty)$. Use calculus to justify your answer. Write NONE if no such value exists.

Solution: Let $h(k) = k - k\ln(k)$ defined on $0.5 \le k \le 2$. To find critical points, we find where $h'(k) = -\ln(k) = 0$. This occurs at $k = 1$. Looking at the table of values

k	0.5	1	2
$h(k)$	$0.5(1 - \ln(0.5)) \approx 0.84$	1	$2(1 - \ln(2)) \approx 0.61$

Then the value of k that maximizes the value of the global minimum of $g(x)$ is $k = 1$.

Answer: $k = 1$.