7. [7 points] Let A and B be positive constants and $f(x) = \dfrac{A(x^2 - B)}{\sqrt{x-3}}$, for $x > 3$. Note that

$$f'(x) = \frac{A(3x^2 - 12x + B)}{2(x-3)^{\frac{3}{2}}} \qquad \text{and} \qquad f''(x) = \frac{3A(x^2 - 8x + 24 - B)}{4(x-3)^{\frac{5}{2}}}.$$

Find all values of A and B so that $f(x)$ has an inflection point at $(8, 2)$. Use calculus to justify that the point $(8, 2)$ is an inflection point. If there are no such values, write NONE.

Solution: In order to have an inflection point at $(8, 2)$ three conditions need to be satisfied:

- $f''(8) = 0$. This is equivalent to

$$0 = \frac{3A((8)^2 - 8(8) + 24 - B)}{4(8-3)^{\frac{5}{2}}}.$$

  This is true if $B = 24$ or $A = 0$. Only $B = 24$ is possible since $A > 0$.
- $f(8) = 2$. This yields

$$2 = \frac{A((8)^2 - B)}{\sqrt{8-3}} = \frac{A(64 - B)}{\sqrt{5}}.$$

  Using $B = 24$ we get $A = \dfrac{\sqrt{5}}{20}$.
- $f''(x)$ needs to have different signs on $(3, 8)$ and $(8, \infty)$.
  - On $(3, 8)$ plug $x = 4$ into $f''(x) = \dfrac{3\sqrt{5}x(x-8)}{80(x-3)^{\frac{5}{2}}}$. We get $f''(4) = \frac{(+)(-)}{+} = -$.
  - On $(8, \infty)$, plug $x = 9$ into $f''(x) = \dfrac{3\sqrt{5}x(x-8)}{80(x-3)^{\frac{5}{2}}}$.We get $f''(9) = \frac{(+)(+)}{+} = +$.

$$A = \frac{\sqrt{5}}{20} \qquad B = 24$$