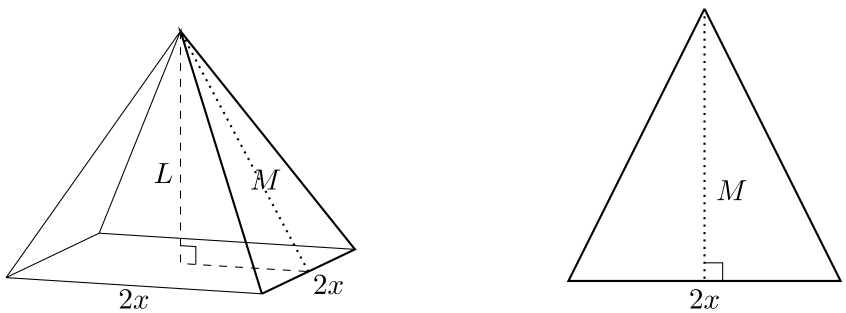
11. [8 points] Jose is building a pyramid-shaped hat with 4 triangular sides of the same shape. Each side has a base of $2x$ centimeters. The height of the hat is L centimeters. Each of the four triangular sides has height M centimeters (see the diagram below).



a. [3 points] Jose plans to use 400 square centimeters of material in the construction of the hat. Find a formula for the height L of the hat only in terms of x. *Your formula should not include the letter M.* Show all your work.

Solution: Using Pythagorean theorem $L = \sqrt{M^2 - x^2}$. Since he uses 400 square centimeters of material in the construction of the hat, we have $4Mx = 400$. This yields

$$L = \sqrt{M^2 - x^2} = \sqrt{\left(\frac{100}{x}\right)^2 - x^2}$$

Answer: $L(x) = \sqrt{\left(\frac{100}{x}\right)^2 - x^2}.$

b. [2 points] The volume of a pyramid is given by $V = \frac{1}{3}Ah$, where A is the area of the base and h is the height of the pyramid. Find a formula for the volume of the hat V, in cubic centimeters, in terms only of the variable x. *Your answer should not include the variables L and/or M.*

Solution: **Answer:** $V(x) = \frac{4}{3}x^2\sqrt{\left(\frac{100}{x}\right)^2 - x^2}$

c. [3 points] What is the domain of the function $V(x)$ in the context of this problem?

Solution: We need to find where $y(x) = \left(\frac{100}{x}\right)^2 - x^2 > 0$ when $x > 0$ (since x is a length). First we find where

$$\left(\frac{100}{x}\right)^2 - x^2 = 0$$
$$\frac{10,000}{x^2} = x^2$$
$$x^4 = 10,000 \qquad \text{this implies} \qquad x = \pm 10.$$

We need to test the sign of $y(x)$ (and hence $V(x)$) on $(0, 10)$ and $(10, \infty)$. The fact that $y(2) = 50^2 - 4 > 0$ and $y(50) = 4 - 50^2 < 0$ shows the domain of $V(x)$ is $0 < x < 10$.

Answer: $0 < x < 10$