5. [7 points] Consider the family of functions $f(x) = bx^5e^{cx}$ with parameters b and c. Note that

$$f'(x) = bx^4e^{cx}(cx+5) \quad \text{and} \quad f''(x) = bx^3e^{cx}(c^2x^2+10cx+20)$$

a. [2 points] Find <u>all</u> values of b and c that make

$$\lim_{x\to\infty} f(x) = \infty \quad \text{AND} \quad \lim_{x\to-\infty} f(x) = 0.$$

Solution: Conditions for b: $b > 0$ Conditions for c: $c > 0$

b. [5 points] Suppose $b > 0$ and $c > 0$. Find the critical point(s) of $f(x)$ and the x-coordinates of the local extrema of $f(x)$. Your answer must be in exact form and may be expressed in terms of the constants b and c. You should use calculus to find and justify your answers. For each answer blank below, write NONE if appropriate.

Solution: There are no points where $f'(x)$ is undefined, so all of the critical points can be found by solving $f'(x) = bx^4e^{cx}(cx+5) = 0$. This yields $x = 0$ and $x = -\frac{5}{c}$. We test the critical point $x = -\frac{5}{c}$ using the Second Derivative Test

$$f''\left(-\frac{5}{c}\right) = b\left(-\frac{5}{c}\right)^3 e^{-5}\left(c^2\left(-\frac{5}{c}\right)^2 + 10c\left(-\frac{5}{c}\right) + 20\right) = 625e^{-5}\frac{b}{c^3} > 0.$$

Hence $x = -\frac{5}{c}$ is a local minimum.

We use the First Derivative Test to classify $x = 0$. We need to test on the intervals $\left(-\frac{5}{c}, 0\right)$ and $(0, \infty)$. We use the points $x = -\frac{4}{c}$ and $x = 1$.

$$f'\left(-\frac{4}{c}\right) = b\left(-\frac{4}{c}\right)^4 e^{-4}\left(c\left(-\frac{4}{c}\right)+5\right) = b\left(\frac{256}{c^4}\right)e^{-4} > 0.$$

$$f'(1) = be^c(c+5) \quad \text{then} \quad f'(1) = +(+)(+) = +.$$

Hence $x = 0$ is neither a local maximum or a local minimum.

Critical point(s) at $x = 0$ and $x = -\frac{5}{c}$

Local max(es) NONE Local min(s) $x = -\frac{5}{c}$