6. [11 points] Ben has recently acquired a cabbage press and is opening a business selling cabbage juice. Let $R(x)$ and $C(x)$ be the revenue and cost, in dollars, of selling and producing x cups of cabbage juice. Ben only has resources to produce up to a hundred cups. After some research, Ben determines that

$$R(x) = 6x - \frac{1}{40}x^2 \qquad \text{for} \qquad 0 \le x \le 100$$

and

$$C(x) = \begin{cases} 60 + 2x & 0 \le x \le 20 \\ 70 + 1.5x & 20 < x \le 100. \end{cases}$$

a. [3 points] What is the smallest quantity of juice Ben will need to sell in order for his profit to not be negative? Round your answer to the nearest hundredth of a cup. Show your work.

Solution: We consider values of x such that $R(x) = C(x)$. We first look in $[0, 20]$

$$60 + 2x = 6x - \frac{1}{40}x^2 \qquad \text{or} \qquad \frac{1}{40}x^2 - 4x + 60 = 0.$$

Using the quadratic formula we get $x = 80 \pm 20\sqrt{10}$. Only one of these two solutions, $x = 80 - 20\sqrt{10} \approx 16.75$, is in the interval $[0, 20]$.

The last step is to verify that $R(x) - C(x)$ is negative on the interval $[0, \quad 80 - 20\sqrt{10})$ and positive on the interval $(80 - 20\sqrt{10}, \quad 20]$. We can test this by picking points in each interval. For example, $R(0) - C(0) = -60$ and $R(20) - C(20) = 10$. **Answer:** 16.75 cups.

For the following parts, determine how many cups of cabbage juice Ben needs to sell in order to maximize the given quantity. If there is no such value, write NONE. Use calculus to find and justify your answers.

b. [3 points] Ben's revenue.

Solution: The critical points of $R(x)$ can be found by solving $R'(x) = 6 - \frac{1}{20}x = 0$. This occurs when $x = 120$ which is not in $[0, 100]$. Hence the maximum has to be at one of the endpoints $x = 0$ or $x = 100$. Since $R(0) = 0$ and $R(100) = 350$, the maximum revenue is attained at $x = 100$. **Answer:** 100 cups.

c. [5 points] Ben's profit.

Solution: Since $P(20) = 10$ and

$$\lim_{x \to 20^-} P(x) = \lim_{x \to 20^-} 6x - \frac{1}{40}x^2 - (60 + 2x) = 10$$

and

$$\lim_{x \to 20^+} P(x) = \lim_{x \to 20^+} 6x - \frac{1}{40}x^2 - (70 + 1.5x) = 10.$$

Then $P(x)$ is continuous on $[0, 20]$. The critical points of $P(x)$ can be found by solving $P'(x) = R'(x) - C'(x) = 0$ in the intervals $(0, 20)$ and $(20, 100)$.

- On $(0, 20)$ we need to solve $6 - \dfrac{1}{20}x = 2$. This yields $x = 80$ (outside the interval).
- On $(20, 100)$ we need to solve $6 - \dfrac{1}{20}x = 1.5$. This yields $x = 90$.

Hence the critical points of $P(x)$ are $x = 0$ and $x = 90$. Since $P(x)$ is continuous then the global maximum must lie on the critical points or in the endpoints.

x	0	20	90	100
$P(x)$	-60	10	132.5	130

Answer: 90 cups.