7. [11 points] Zavier the zoo-keeper is breeding fish in a large aquarium. As the population of fish increases, he notices that the amount of waste in the bottom of aquarium is also increasing. Each week he measures how much waste has accumulated at the bottom of the aquarium. The function $W(t)$ models the amount of waste, in millimeters, at the bottom of the aquarium t weeks after Zavier began his measurements. Note the following information about the function $W(t)$.

- $W(t)$ is continuous on the interval $[0, 9]$.
- During the first 3 weeks, the amount of waste increases exponentially from 102.4 mm at time $t = 0$ to 200 mm at time $t = 3$.
- After 3 weeks, Zavier buys several catfish to eat the waste at the bottom of the aquarium. Over the next 6 weeks, the **rate of change** in the amount of waste (in millimeters per week) is given by the function $g(t) = t^2 - 12t + 26$.

a. [8 points] Write a piecewise defined formula for the **continuous** function $W(t)$ on the interval $[0, 9]$. Show all your work.

Solution:

- On $0 \le t \le 3$: We have $W(t) = ab^t$ with $W(0) = 102.4$ and $W(3) = 200$. Hence $a = 102.4$ and $W(3) = 102.4b^3 = 200$. Solving for b we get $b = \left(\frac{200}{102.4}\right)^{\frac{1}{3}}$. Then $W(t) = 102.4\left(\frac{200}{102.4}\right)^{\frac{1}{3}t}$ in $0 \le t \le 3$.
- On $3 < t \le 9$: Since $W'(t) = g(t)$, then $W(t) = \frac{1}{3}t^3 - 6t^2 + 26t + C$ where C is a constant that we have to determine. We know that $W(t)$ is continuous at $t = 3$, then

$$\lim_{t\to 3^+} W(t) = \lim_{t\to 3^+} \frac{1}{3}t^3 - 6t^2 + 26t + C = 33 + C = W(3) = 200.$$

This yields that $C = 200 - 33 = 167$. Hence

$$W(t) = \begin{cases} 102.4\left(\dfrac{200}{102.4}\right)^{\frac{1}{3}t} & \text{if } \quad 0 \le t \le 3 \\ \dfrac{1}{3}t^3 - 6t^2 + 26t + 167 & \text{if } \quad 3 < t \le 9. \end{cases}$$

b. [3 points] Zavier continues recording data and notices that, after 50 weeks, the amount of waste is modeled by the function

$$Q(t) = \frac{4(a-t)^2(bt-2)^2}{(3t^2+7t+10)(4-t)(-2t+c)}$$

where a, b, and c are positive constants. What happens to the amount of waste in the long run? Circle the correct answer and fill in the blank if necessary. Your answer may include the constants a, b or c.

Solution:

i) It increases without limit.
ii) It approaches zero.
iii) $\boxed{\text{It approaches a positive limit with value } L \text{ where } L = \frac{2b^2}{3}}$
iv) None of these.