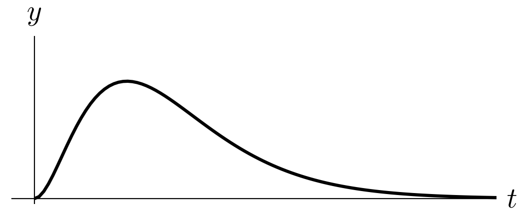
4. [7 points] Lin inflates a balloon using a helium pump. When she turns off the pump, the balloon immediately begins to deflate. Lin believes that she can model the balloon's volume, in cubic feet (ft^3), by the function

$$V(t) = \frac{at^2}{e^{bt}},$$

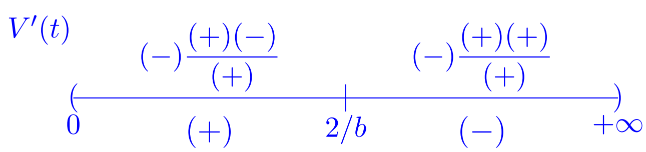
where t is the time, in seconds, after she begins inflating the balloon, and where a and b are positive constants. As an example, this function is shown to the right for one choice of the constants a and b. Note that the derivative of $V(t)$ is given by



$$V'(t) = -\frac{at(bt-2)}{e^{bt}}.$$

a. [4 points] The function $V(t)$ appears to have a local maximum at some time $t > 0$. Find the time at which this local maximum occurs. Use calculus to find your answer, and be sure to give enough evidence that the point you find is indeed a local maximum. Your answer may be in terms of a and/or b.

Solution: The critical points of $V(t)$ are 0 and $2/b$, and since b is positive, we know that $2/b > 0$. Below is a table showing the signs of $V'(t)$ for $t > 0$, with sign logic to justify how we know when $V'(t)$ is positive and when it is negative. Note: a and e^{bt} are always positive.



By the first derivative test, $t = 2/b$ is a local maximum. (The second derivative test can also be used.)

Answer: local max at $t =$ $2/b$

b. [3 points] Lin knows that it took 8 seconds to inflate the balloon, and that its volume at that time was 1.5 ft^3. Find the **exact** values of a and b for Lin's model. Show your work.

Solution: If Lin took 8 seconds to inflate the balloon, that means the local maximum we found in part (a) needs to occur at $t = 8$. If $2/b = 8$, then $b = 1/4$. The volume of the balloon at this time was 1.5 ft^3, which means

$$1.5 = V(8) = \frac{a(8)^2}{e^{(1/4)(8)}}.$$

Solving for a, we get $a = 1.5e^2/64$.

Answer: $a = \frac{1.5e^2}{64}$ and $b =$ $1/4$

