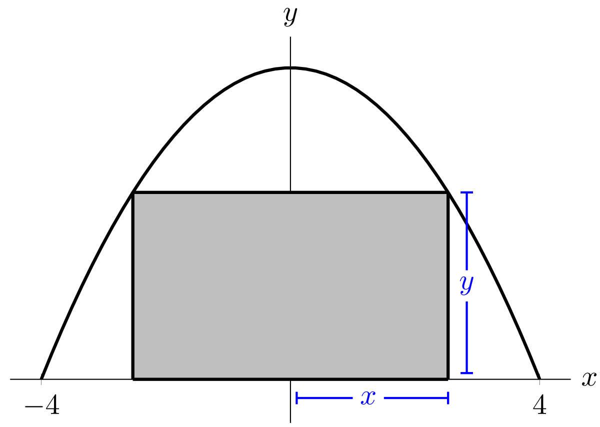
6. [8 points]

A garden store plans to build a large rectangular sign on the interior wall at one end of their greenhouse. For x and y in meters, the curved roof of the greenhouse is described by the function

$$y = 5 - \frac{5}{16}x^2 \text{ for } -4 \le x \le 4.$$

This curve is graphed to the right; the shaded rectangle is one possible sign that could be built.



Find the width and height of the sign with the maximum area. Use calculus to find your answers, and be sure to show enough evidence that the values you find do in fact maximize the area.

Solution: Consider the rectangular sign labeled as in the above picture, with a width of $2x$ and height of y. The area of this sign is $A = (2x)(y)$. We also know that y can be expressed in terms of x, namely,

$$y = 5 - \frac{5}{16}x^2.$$

After making this substitution, the area of the sign is a function of x:

$$A(x) = (2x)\left(5 - \frac{5}{16}x^2\right) = 10x - \frac{5}{8}x^3.$$

The domain of this function is $[0, 4]$. To optimize, we need to find the critical points of $A(x)$ on its domain:

$$A'(x) = 10 - \frac{15}{8}x^2, \quad \text{so} \quad A'(x) = 0 \quad \text{when} \quad 10 = \frac{15}{8}x^2.$$

The critical points can be found by solving

$$\frac{15}{8}x^2 = 10 \quad \text{simplifies to} \quad x^2 = \frac{80}{15}, \quad \text{so there are two values:} \quad x = \pm\sqrt{\frac{80}{15}} \approx \pm 2.3094.$$

The value of $x \approx -2.3094$ is outside our domain of $[0, 4]$, so the only critical point we need to consider is $x \approx 2.3094$. We know that neither of endpoints of the interval $[0, 4]$ are the global maximum, since $A(0) = A(4) = 0$ but $A(2.3094) \approx 15.396 > 0$. Therefore, because there is only one critical point, it must be the global maximum.

Now, the width of the sign is $2x$, not x, so the width that maximizes area is $2\sqrt{80/15} \approx 4.6188$. The corresponding height is

$$y = 5 - \frac{5}{16}\left(\sqrt{\frac{80}{15}}\right)^2 = \frac{10}{3}.$$

Answers: area is maximized when width = $2\sqrt{\frac{80}{15}} \approx 4.6188$ meters

and height = $\frac{10}{3} \approx 3.3333$ meters