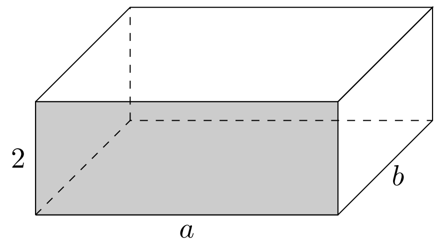
5. [11 points] A museum is building a specialized box to display a new exhibit. The box needs to have a volume of 20 cubic feet and a height of 2 feet. The front of the box, which is shaded in the diagram below, will be made of glass, which costs \$4 per square foot. The top, sides, back, and bottom of the box will be made of a metal that costs \$1 per square foot. Let a and b be the length and width, in feet, of the box, as shown below.



What values of a and b will minimize the cost of the box, and what will the cost be in that case? Use calculus to find and justify your answer, and be sure to show enough evidence that the values you find do in fact minimize the cost of the box.

Solution:
The cost for the box is

$$C = 4 \cdot 2a + ab + ab + 2a + 2b + 2b$$
$$= 10a + 2ab + 4b.$$

Solving for a:

$$20 = 2ab \text{ , so } a = \frac{10}{b}$$

Therefore the cost function is

$$C(b) = \frac{10 \cdot 10}{b} + \frac{2 \cdot 10b}{b} + 4b$$
$$= \frac{100}{b} + 20 + 4b$$
$$C'(b) = -\frac{100}{b^2} + 4$$

is zero when $b^2 = 25$ or $b = 5$.
Now $C(5) = 60$ while

$$\lim_{b \to 0^+} C(a) = \infty \text{ and } \lim_{b \to \infty} C(a) = \infty,$$

so $b = 5$ is the **global max**, with $a = \frac{10}{5} = 2$.

Answer: the cost is minimized when $a =$ 2 and $b =$ 5

and the minimum cost is \$ 60