7. [9 points] Consider the family of functions

\[ m(x) = x + \frac{c^2}{x} \]

defined for \( x > 0 \), where \( c \) is a positive constant.

Throughout this problem, use calculus to find your answers, show all your work, and be sure to show enough evidence to justify your conclusions.

a. [2 points] Any function in this family has only one critical point on its domain \( x > 0 \). In terms of \( c \), what is the \( x \)-coordinate of this critical point?

Solution: \( m'(x) = 1 - \frac{c^2}{x^2} \).

\( m'(x) \) DNE at \( x = 0 \) which is not in the domain.

\( m'(x) = 0 \) at \( x = c \) which is in the domain.

Answer: \( c \) or \( x = c \)

b. [3 points] Is the critical point a local minimum, a local maximum, neither, or is there not enough information to decide? Circle your answer below.

Solution:

For the 1st derivative test: \( m'(x) = 1 - \frac{c^2}{x^2} \) and \( m'(2c) = 1 - \frac{c^2}{2^2} = \frac{3}{4} > 0 \). Therefore \( x = c \) is a local min.

For the 2nd derivative test: \( m''(x) = 0 + 2 \frac{c^2}{x^3} \) and \( m''(c) = \frac{2}{c} > 0 \). Therefore \( x = c \) is a local min.

Answer: local min

neither

not enough info

c. [2 points] Find the \( x \)-coordinates of all inflection points of \( m(x) \), or if there are none, write NONE.

Solution: \( m''(x) = 0 + 2 \frac{c^2}{x^3} \) which is defined everywhere in the domain and not equal to zero on the domain. Therefore \( m(x) \) has no inflection points.

Answer: Inflection point(s) at \( x = \) NONE

d. [2 points] Find the value for \( c \) such that \( m(x) = 10 \) at its critical point.

Solution: The value at \( x = c \) is \( m(c) = c + \frac{c^2}{c} = 2c \). If it is equal to 10, then \( 2c = 10 \), or \( c = 5 \).

Answer: \( c = 5 \)