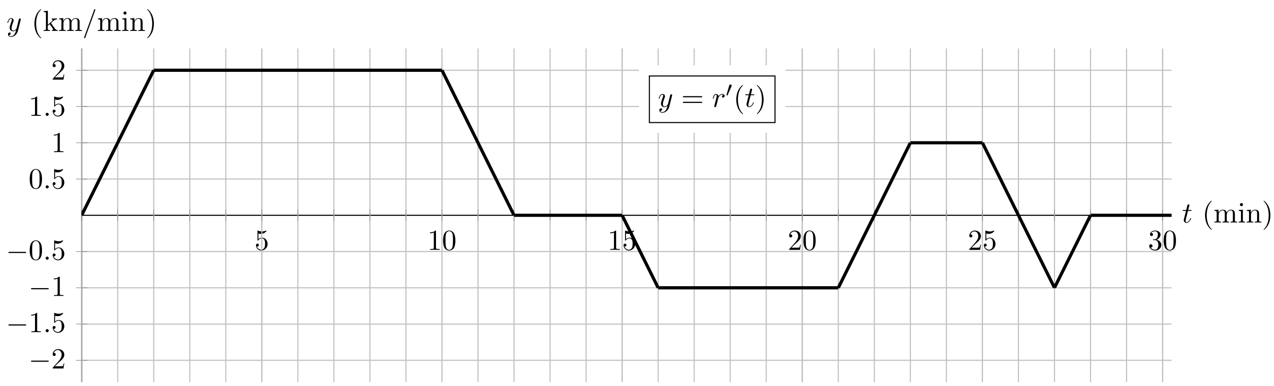
6. [15 points] Caroline, an amateur astronomer, is driving at night along a straight road through the desert between the small towns of Tycho and Brahe, trying to find the darkest spot between them in order to obtain the best viewing conditions. Brahe is 37 kilometers (km) east of Tycho.

Let $r(t)$ be Caroline's position along the road, in kilometers east of Tycho, t minutes after she departs Tycho at 9pm. At 9:28pm, she decides she is close enough to the darkest spot, and she parks her car and sets up her telescope. Pictured below is a graph of $r'(t)$, the **derivative** of $r(t)$.



a. [1 point] How many times did Caroline turn around and retrace part of her path, before eventually coming to a stop?

Answer: She turned around **3** times.

b. [2 points] Given that Tycho and Brahe are 37 km apart, what was the closest Caroline came to Brahe? That is, what was her *minimum distance from Brahe*, over the course of her drive?

Solution: Caroline drove $\int_0^{12} r'(t)\,dt = 20$ km east, then $-\int_{15}^{22} r'(t)\,dt = 6$ km west, followed by $\int_{22}^{26} r'(t)\,dt = 3$ km east and $-\int_{26}^{28} r'(t)\,dt = 1$ km west. It follows that she was furthest east (i.e., closest to Brahe) when she turned around for the first time, 20 km east of Tycho, which is 17 km west of Brahe.

Answer: The closest Caroline came to Brahe was **17** kilometers.

c. [2 points] The road Caroline is driving on crosses railroad tracks exactly 10 km east of Tycho. At what time did Caroline first cross these railroad tracks?

Solution: We must find the least $b > 0$ such that $\int_0^b r'(t)\,dt = 10$. By counting squares (or adding up areas of triangles and rectangles), we see that $b = 6$. So Caroline crossed the tracks 6 minutes after 9pm.

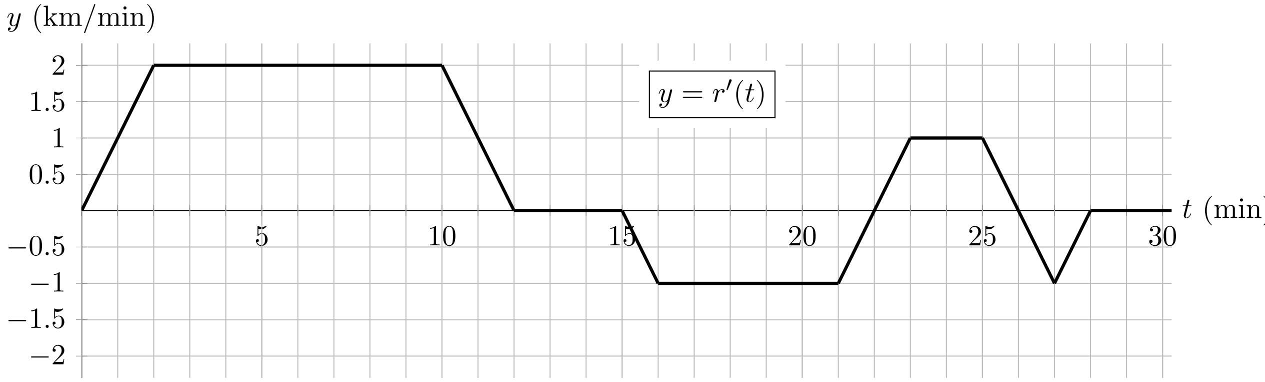
Answer: She first crossed the tracks at **9:06 pm** .

d. [2 points] Write an expression *involving one or more integrals* for the total distance in kilometers that Caroline traveled while searching for the darkest spot.

Answer: $\int_0^{28} |r'(t)|dt$, **or** $\int_0^{12} r'(t)dt - \int_{15}^{22} r'(t)dt + \int_{22}^{26} r'(t)dt - \int_{26}^{28} r'(t)dt$

This problem continues on the next page.

This problem continues from the previous page. The graph of $r'(t)$ is displayed again for convenience. Recall that Brahe is 37 kilometers east of Tycho.



After a while, Caroline realizes that she could have found the darkest spot between Tycho and Brahe *exactly* by solving an optimization problem, since the apparent brightness of a light source is directly proportional to the light's brightness and inversely proportional to the square of the observer's distance from the light source.

e. [3 points] Assuming that Brahe is twice as bright as Tycho, and that no other towns or light sources are near enough to be significant, find a function $f(x)$ that models, in appropriate units, the apparent brightness of the two towns at a point x km east of Tycho.

Note: you do not need to minimize $f(x)$, or give units.

Solution: Since Brahe is twice as bright as Tycho, there is some positive constant c such that, in appropriate units, the apparent brightness of Tycho x km east of Tycho is $\frac{c}{x^2}$ and the apparent brightness of Brahe x km east of Tycho is $\frac{2c}{(37-x)^2}$, so the total apparent brightness of the two towns at that point is

$$f(x) = \frac{c}{x^2} + \frac{2c}{(37-x)^2}.$$

This is a correct answer, although the proportionality constant can be eliminated by changing units, and does not affect the minimum anyway, so one could also answer more simply:

Answer: $f(x) = \frac{1}{x^2} + \frac{2}{(37-x)^2}$

f. [1 point] In order to find the darkest point, over what domain should the function $f(x)$ be minimized?

Answer: $[0, 37]$

g. [4 points] Caroline uses calculus to minimize $f(x)$, and finds the darkest point between the towns to be exactly d km east of Tycho. Happily, this point turns out to be just 0.4 km east of where she actually parked. Letting $g(t) = f(r(t))$, determine the *signs* of the quantities below by clearly writing $<$, $=$, or $>$ in the given boxes.

i. $f'(d)$ $\boxed{=}$ 0 ii. $g'(3)$ $\boxed{<}$ 0 iii. $g'(11)$ $\boxed{>}$ 0

iv. $g'(14)$ $\boxed{=}$ 0 v. $g'(16)$ $\boxed{<}$ 0 vi. $g'(22)$ $\boxed{=}$ 0

Solution: Note that $f'(d) = 0$ because d is a local extremum of f. Furthermore, $g'(t) = 0$ whenever Caroline's velocity, $r'(t)$, is zero, and $g'(t)$ is negative whenever Caroline is moving *towards* the darkest spot, and positive whenever she is moving *away* from it.