8. [11 points] Consider the family of functions $f(x) = x^4e^{-cx}$, where $c > 0$. Note that
\[
f'(x) = x^3e^{-cx}(4 - cx) \quad \text{and} \quad f''(x) = c^2x^2e^{-cx}(x - \frac{2}{c})(x - \frac{6}{c}).
\]

a. [2 points] List all the critical points of $f(x)$ and $f'(x)$, in terms of $c$. No justification necessary.

**Answer:** Critical points of $f(x)$: \(0, \frac{4}{c}\). Critical points of $f'(x)$: \(0, \frac{2}{c}, \frac{6}{c}\).

b. [4 points] Determine, in terms of the parameter $c$, the intervals of concavity of the function $f(x)$. Use calculus to find your answers, show all your work, and be sure to show enough evidence to justify your conclusions.

**Solution:** The function $f(x)$ will be concave up on intervals where $f''(x) > 0$, and concave down on intervals where $f''(x) < 0$. Noting that $f''(x) = 0$ at $x = 0, \frac{2}{c}, \frac{6}{c}$, we will make a sign chart for $f''(x)$ where we break up our number line at these three points and determine the sign of $f''(x)$ in each of the resulting intervals. Note that $c^2x^2e^{-cx} > 0$ for all $x \neq 0$, and that $(x - \frac{2}{c}) > 0$ when $x > \frac{2}{c}$, while $(x - \frac{6}{c}) > 0$ when $x > \frac{6}{c}$. This gives us the table of signs:

<table>
<thead>
<tr>
<th>Interval</th>
<th>$c^2x^2e^{-cx}$</th>
<th>$(x - \frac{2}{c})$</th>
<th>$(x - \frac{6}{c})$</th>
<th>$f''(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 0)$</td>
<td>$+$</td>
<td>$-$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>$(0, \frac{2}{c})$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td>$(\frac{2}{c}, \frac{6}{c})$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$(\frac{6}{c}, \infty)$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Or, equivalently, the following sign chart:

$$f''(x): \begin{cases} + \cdot \cdot \cdot = & + \cdot \cdot \cdot = + \cdot \cdot \cdot = - \cdot \cdot \cdot = + \cdot \cdot \cdot = + \\ \cup & \quad 0 \quad \cup \quad \frac{2}{c} \quad \cup \quad \frac{6}{c} \quad \cup \end{cases}$$

**Answer:** Intervals on which $f(x)$ is concave up: \((-\infty, \frac{2}{c}) \text{ and } (\frac{6}{c}, \infty)\)

**Answer:** Intervals on which $f(x)$ is concave down: \(\left(\frac{2}{c}, \frac{6}{c}\right)\)

c. [2 points] Circle the $x$-coordinates of all inflection points of $f(x)$ in terms of the parameter $c$ that are listed below, or, if no inflection points of $f(x)$ are listed below, circle NONE.

$$0, e^{-c}, e^c, \frac{2}{c}, \frac{4}{c}, \frac{6}{c}, \text{ NONE}$$

d. [3 points] For each $c > 0$, the function $y = f(x)$ has exactly one local extremum in $(0, \infty)$.

i. Find the unique value of $c$ such that $f(x)$ has a local extreme value of $y = 1$ in the interval $(0, \infty)$. Show your work.

**Solution:** We are given that $f(x)$ has exactly one local extremum in $(0, \infty)$, so this extremum must occur at the only critical point $f(x)$ has in $(0, \infty)$, namely $x = \frac{4}{c}$. So we set $f\left(\frac{4}{c}\right) = 1$ and solve for $c$. We get

$$1 = f\left(\frac{4}{c}\right) = \left(\frac{4}{c}\right)^4e^{-4}, \quad \text{so} \quad e^4 = \left(\frac{4}{c}\right)^4, \quad \text{which means} \quad e = \frac{4}{c}.$$  

It follows that $c = \frac{4}{e}$. Since $f''\left(\frac{4}{c}\right) < 0$, this is a local MAX by the Second Derivative Test.

**Answer:** $c = \frac{4}{e}$

ii. Is the local extremum of $f(x)$ in $(0, \infty)$ a max or a min? Circle your answer: **MAX** MIN

(No justification is necessary.)