

8. [11 points] Consider the family of functions $f(x) = x^4 e^{-cx}$, where $c > 0$. Note that

$$f'(x) = x^3 e^{-cx}(4 - cx) \quad \text{and} \quad f''(x) = c^2 x^2 e^{-cx} \left(x - \frac{2}{c}\right) \left(x - \frac{6}{c}\right).$$

a. [2 points] List all the critical points of $f(x)$ and $f'(x)$, in terms of c . *No justification necessary.*

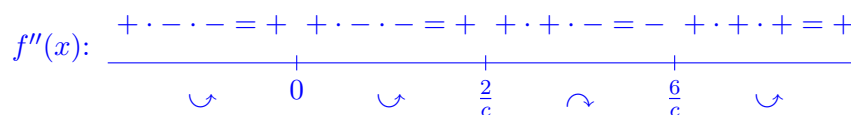
Answer: Critical points of $f(x)$: 0, $\frac{4}{c}$. Critical points of $f'(x)$: 0, $\frac{2}{c}$, $\frac{6}{c}$.

b. [4 points] Determine, in terms of the parameter c , the intervals of concavity of the function $f(x)$. Use calculus to find your answers, show all your work, and be sure to show enough evidence to justify your conclusions.

Solution: The function $f(x)$ will be concave up on intervals where $f''(x) > 0$, and concave down on intervals where $f''(x) < 0$. Noting that $f''(x) = 0$ at $x = 0, \frac{2}{c}, \frac{6}{c}$, we will make a sign chart for $f''(x)$ where we break up our number line at these three points and determine the sign of $f''(x)$ in each of the resulting intervals. Note that $c^2 x^2 e^{-cx} > 0$ for all $x \neq 0$, and that $(x - \frac{2}{c}) > 0$ when $x > \frac{2}{c}$, while $(x - \frac{6}{c}) > 0$ when $x > \frac{6}{c}$. This gives us the table of signs:

	$(-\infty, 0)$	$(0, \frac{2}{c})$	$(\frac{2}{c}, \frac{6}{c})$	$(\frac{6}{c}, \infty)$
$c^2 x^2 e^{-cx}$	+	+	+	+
$(x - \frac{2}{c})$	-	-	+	+
$(x - \frac{6}{c})$	-	-	-	+
$f''(x)$	+	+	-	+

Or, equivalently, the following sign chart:



Answer: Intervals on which $f(x)$ is concave up: $(-\infty, \frac{2}{c})$ and $(\frac{6}{c}, \infty)$

Answer: Intervals on which $f(x)$ is concave down: $(\frac{2}{c}, \frac{6}{c})$

c. [2 points] Circle the x -coordinates of all inflection points of $f(x)$ in terms of the parameter c that are listed below, or, if no inflection points of $f(x)$ are listed below, circle NONE.

0 e^{-c} e^c $\frac{2}{c}$ $\frac{4}{c}$ $\frac{6}{c}$ NONE

d. [3 points] For each $c > 0$, the function $y = f(x)$ has exactly one local extremum in $(0, \infty)$.

i. Find the unique value of c such that $f(x)$ has a local extreme value of $y = 1$ in the interval $(0, \infty)$. *Show your work.*

Solution: We are given that $f(x)$ has exactly one local extremum in $(0, \infty)$, so this extremum must occur at the only critical point $f(x)$ has in $(0, \infty)$, namely $x = \frac{4}{c}$. So we set $f(\frac{4}{c}) = 1$ and solve for c . We get

$$1 = f\left(\frac{4}{c}\right) = \left(\frac{4}{c}\right)^4 e^{-4}, \quad \text{so} \quad e^4 = \left(\frac{4}{c}\right)^4, \quad \text{which means} \quad e = \frac{4}{c}.$$

It follows that $c = \frac{4}{e}$. Since $f''\left(\frac{4}{c}\right) < 0$, this is a local MAX by the Second Derivative Test.

Answer: $c =$ $\frac{4}{e}$

ii. Is the local extremum of $f(x)$ in $(0, \infty)$ a max or a min? Circle your answer: MAX MIN
(*No justification is necessary.*)