8. [11 points] Consider the family of functions $f(x) = x^4 e^{-cx}$, where $c > 0$. Note that

$$f'(x) = x^3 e^{-cx}(4 - cx) \qquad \text{and} \qquad f''(x) = c^2 x^2 e^{-cx}\left(x - \tfrac{2}{c}\right)\left(x - \tfrac{6}{c}\right).$$

a. [2 points] List all the critical points of $f(x)$ and $f'(x)$, in terms of c. *No justification necessary.*

Answer: Critical points of $f(x)$: $\mathbf{0,\ \frac{4}{c}}$. Critical points of $f'(x)$: $\mathbf{0,\ \frac{2}{c},\ \frac{6}{c}}$.

b. [4 points] Determine, in terms of the parameter c, the intervals of concavity of the function $f(x)$. Use calculus to find your answers, show all your work, and be sure to show enough evidence to justify your conclusions.

Solution: The function $f(x)$ will be concave up on intervals where $f''(x) > 0$, and concave down on intervals where $f''(x) < 0$. Noting that $f''(x) = 0$ at $x = 0, \frac{2}{c}, \frac{6}{c}$, we will make a sign chart for $f''(x)$ where we break up our number line at these three points and determine the sign of $f''(x)$ in each of the resulting intervals. Note that $c^2x^2e^{-cx} > 0$ for all $x \neq 0$, and that $(x - \frac{2}{c}) > 0$ when $x > \frac{2}{c}$, while $(x - \frac{6}{c}) > 0$ when $x > \frac{6}{c}$. This gives us the table of signs:

	$(-\infty, 0)$	$(0, \frac{2}{c})$	$(\frac{2}{c}, \frac{6}{c})$	$(\frac{6}{c}, \infty)$
$c^2x^2e^{-cx}$	$+$	$+$	$+$	$+$
$(x - \frac{2}{c})$	$-$	$-$	$+$	$+$
$(x - \frac{6}{c})$	$-$	$-$	$-$	$+$
$f''(x)$	$+$	$+$	$-$	$+$

Or, equivalently, the following sign chart:

$f''(x)$: $+\cdot-\cdot-=+$ | 0 | $+\cdot-\cdot-=+$ | $\frac{2}{c}$ | $+\cdot+\cdot-=-$ | $\frac{6}{c}$ | $+\cdot+\cdot+=+$

(concave up on $(-\infty,0)$, concave up on $(0,\frac{2}{c})$, concave down on $(\frac{2}{c},\frac{6}{c})$, concave up on $(\frac{6}{c},\infty)$)

Answer: Intervals on which $f(x)$ is concave up: $\left(-\infty, \frac{2}{c}\right)$ **and** $\left(\frac{6}{c}, \infty\right)$

Answer: Intervals on which $f(x)$ is concave down: $\left(\frac{2}{c}, \frac{6}{c}\right)$

c. [2 points] Circle the x-coordinates of all inflection points of $f(x)$ in terms of the parameter c that are listed below, or, if no inflection points of $f(x)$ are listed below, circle NONE.

0 $\qquad e^{-c}$ $\qquad e^{c}$ $\qquad \boxed{\frac{2}{c}}$ $\qquad \frac{4}{c}$ $\qquad \boxed{\frac{6}{c}}$ $\qquad$ NONE

d. [3 points] For each $c > 0$, the function $y = f(x)$ has exactly one local extremum in $(0, \infty)$.

i. Find the unique value of c such that $f(x)$ has a local extreme value of $y = 1$ in the interval $(0, \infty)$. *Show your work.*

Solution: We are given that $f(x)$ has exactly one local extremum in $(0, \infty)$, so this extremum must occur at the only critical point $f(x)$ has in $(0, \infty)$, namely $x = \frac{4}{c}$. So we set $f\left(\frac{4}{c}\right) = 1$ and solve for c. We get

$$1 = f\left(\tfrac{4}{c}\right) = \left(\tfrac{4}{c}\right)^4 e^{-4}, \qquad \text{so} \quad e^4 = \left(\tfrac{4}{c}\right)^4, \qquad \text{which means} \quad e = \tfrac{4}{c}.$$

It follows that $c = \frac{4}{e}$. Since $f''\left(\frac{4}{c}\right) < 0$, this is a local MAX by the Second Derivative Test.

Answer: $c = 4/e$

ii. Is the local extremum of $f(x)$ in $(0, \infty)$ a max or a min? Circle your answer: $\boxed{\text{MAX}}$ MIN

(No justification is necessary.)