6. [10 points] Let $g(x)$ be the function defined by

$$g(x) \;=\; x\sin(\pi x) + \frac{1}{\pi}\cos(\pi x).$$

The derivative of $g(x)$ is

$$g'(x) \;=\; \pi x\cos(\pi x).$$

As a reminder, $1 = \cos(0) = \sin(\frac{\pi}{2})$, and $-1 = \sin(-\frac{\pi}{2}) = \cos(\pi) = \cos(-\pi)$.

a. [3 points] Find all critical points of the function $g(x)$ that are in the interval $[-\frac{1}{2}, 1]$.

Solution: We need to find points in $[-\frac{1}{2}, 1]$ such that

$$g'(x) = \pi x\cos(\pi x) = 0.$$

We get $x = 0$ or $\cos(\pi x) = 0$.
Next, $\cos^2(\pi x) = 1 - \sin^2(\pi x) = 0$, so $\sin(\pi x) = \pm 1$. Now, from $\sin(-\frac{\pi}{2}) = -1$ and $\sin(\frac{\pi}{2}) = 1$ it follows that $x = \pm 1/2$.

Answer: $x =$ $-1/2$, 0, $1/2$

b. [5 points] Find all x-values where the global extrema of $g(x)$ occur on the interval $[-\frac{1}{2}, 1]$. Be sure to show your work and justify your answers.

Solution: Let us compute the values of g at the critical points and end points:

$$\begin{aligned} g(-1/2) &= -\frac{1}{2}\sin(-\pi/2) + \frac{1}{\pi}\cos(-\pi/2) = 1/2, \\ g(0) &= \frac{1}{\pi}\cos(0) = 1/\pi, \\ g(1/2) &= \frac{1}{2}\sin(\pi/2) + \frac{1}{\pi}\cos(\pi/2) = 1/2, \\ g(1) &= \sin(\pi) + \frac{1}{\pi}\cos(\pi) = -1/\pi. \end{aligned}$$

We note that $\pi > 2$, so $1/\pi < 1/2$.

Answer: The maximum occurs at $x =$ $-1/2$, $1/2$

Answer: The minimum occurs at $x =$ 1

c. [2 points] Find a formula for the linear approximation $L(x)$ of the function $g(x)$ at the point $\big(-2, \frac{1}{\pi}\big)$.

Solution:

$$L(x) \;=\; g(-2) + g'(-2)\cdot(x+2) \;=\; \frac{1}{\pi} - 2\pi\cos(-2\pi)\cdot(x+2) \;=\; \frac{1}{\pi} - 2\pi(x+2)$$

Answer: $L(x) =$ $1/\pi - 2\pi(x+2)$