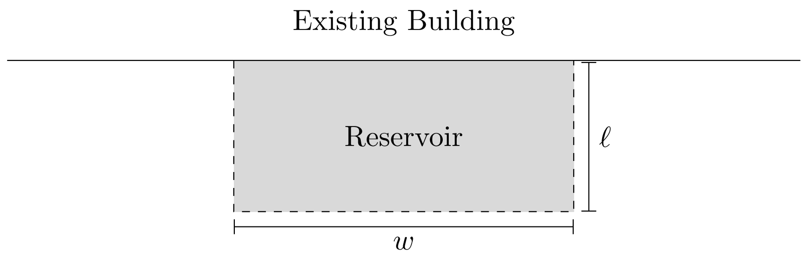
4. [14 points]
A city is building a new water treatment facility. A water reservoir will be built next to an existing building as shown here (viewed from above):



Three walls of the reservoir (dashed) and the floor of the reservoir (shaded) will be made of concrete. The cost for the concrete on the three walls is \$500 per meter of perimeter and the cost for the concrete for the floor is \$1000 per square meter.

a. [3 points] Find a formula for the cost C for the concrete to build the reservoir if it has dimensions ℓ and w (both in meters) as shown above.

Answer: $C =$ $500(2\ell + w) + 1000\ell w$

b. [2 points] If the budget for the total cost of the concrete is \$5,000,000, write a formula for the length ℓ of the reservoir in terms of its width w. This relationship can be used to find the largest reservoir possible, given the concrete budget, but you do not need to find this.

Solution: Using our formula from part (a) we get the following relationship between ℓ and w:

$$5,000,000 = 500(2\ell + w) + 1000\ell w.$$

Solving for ℓ we get:

$$10,000 = 2\ell + w + 2\ell w$$

$$10,000 = 2\ell(w + 1) + w$$

$$\ell = \frac{10,000 - w}{2(w + 1)}$$

Answer: $\ell =$ $\frac{10,000 - w}{2(w+1)}$

4. (continued) The city is also looking for the most cost-effective way to build a large tank in the shape of a cylinder for the water treatment facility. The cost, in dollars, of a tank with radius r and height h (both given in meters) is given by

$$50\pi r^2 + 80\pi rh.$$

c. [2 points] The tank must have volume 10,000 cubic meters, so that $\pi r^2 h$ =10,000. Given this constraint, find a formula for the cost $T(r)$ of the tank, in dollars, that is a function of the variable r only. *Your answer should not include the height h of the tank.*

Answer: $T(r) =$ $50\pi r^2 + \frac{80\cdot 10{,}000}{r}$

d. [2 points] Note that, in context, $T(r)$ has a domain of $(0, \infty)$. Determine $\lim_{r\to 0^+} T(r)$ and $\lim_{r\to\infty} T(r)$. Each answer should either be a number, or ∞ or $-\infty$.

$\lim_{r\to 0^+} T(r) =$ ∞ $\lim_{r\to\infty} T(r) =$ ∞

e. [5 points] Find the radius that will minimize the cost of the tank. Use calculus, and show your work, but you need not simplify your numerical answers. *Make sure to justify why your answer is the global minimum; you may use work from previous parts of this problem.*

Solution: Setting $T'(r) = 100\pi r - \frac{80\cdot 10{,}000}{r^2} = 0$, we find one positive critical point of $(8000/\pi)^{1/3}$. This critical point is a local minimum by the 2nd derivative test since $T''(r) = 100\pi + \frac{160\cdot 10{,}000}{r^3}$ is always positive. Since this is the only critical point, and since by part b the cost grows infinitely large at both ends of the domain, it must be the global minimum.

Answer: Cost is minimized when $r =$ $20/\sqrt[3]{\pi}$