(10.) Cost and revenue functions for a charter bus company are shown in the figure below, where \( q \) is the number of buses that the company owns.

(a) (4 pts) Should the company add a 50\(^{th}\) bus? How about a 100\(^{th}\)? Explain your answers using marginal revenue and marginal cost. (You may illustrate your reasons graphically as well, if you like.)

The company should add the 50\(^{th}\) bus because at \( q = 50 \) \( MC \) (on the slope of \( R(q) \)) is less than \( MR \) – i.e., revenue is increasing faster than cost. The company should not add the 100\(^{th}\) bus because at \( q = 100 \), costs are increasing at a faster rate than revenue – i.e., \( MR < MC \).

(b) (3 pts) What does \( C'(50) = A \) (\( A \) a constant) mean in the context of this problem? What are the units of the 50 and the units of \( A \)?

In the expression \( C'(50) = A \), the units of 50 are buses, while units of \( A \) are dollars per bus. The expression represents the rate at which costs are increasing, once the company has 50 buses. Approximately the cost of going from 50 - 51 buses is \( DA \).

(c) (4 pts) Estimate the number of buses the company should have in order to maximize profit. Explain how you determined your estimate.

Profit will be maximized when \( R(q) - C(q) \) is greatest – which also (by calculus) occurs when \( R'(q) = C'(q) \). This means the slope of the tangent to \( C(q) \) is equal to the slope of \( R(q) \). This happens to be around \( q = 80 \).
(Problem 10 continued)

(d) (6 pts)

(i) If the average cost, \( a(q) \), is given by \( a(q) = \frac{C(q)}{q} \), approximate \( q_0 \) so that \( a(q_0) \) is the minimal average cost.

From the graph, minimum average cost appears to be when \( q = 60 \).

(ii) Show *analytically* that average cost will be minimized when \( C'(q) = a(q) \).

Given \( a(q) = \frac{C(q)}{q} \), then

\[
\frac{d}{dq} \left( \frac{C(q)}{q} \right) = \frac{C'(q)q - C(q)}{q^2} = \frac{C'(q)}{q} - a(q) = 0.
\]

\[
\Rightarrow \frac{C'(q)}{q} = a(q) = \frac{C(q)}{q^2}.
\]

Note: \( \frac{C(q)}{q^2} \) can be visualized as slope from \((0,0)\) to \((q, C(q))\). When slopes decrease for \( 0 < q < q_0 \) and increase for \( q > q_0 \), then there is a \( \text{Min} @ q = q_0 \).

(iii) Demonstrate on the graph below how this result can be shown graphically.

(11.) And, for good measure, one last derivative.... No need to simplify, but show all your work.

(3 pts) Find the derivative of \( k(t) = \frac{(3t - 4)}{\cos(2t)} \).

\[
k'(t) = \frac{\cos(2t)(3) - (3t - 4)(-\sin(2t))2}{\cos^2(2t)} = \frac{3\cos(2t) + (6t - 8)\sin(2t)}{\cos^2(2t)}.
\]