(12.) The city council is planning to construct a new sports ground in the shape of a rectangle with semicircular ends. A running track 400 meters long is to go around the perimeter.

(a) (6 pts) What choice of dimensions will make the rectangular area in the center as large as possible?

Given:
$P = 2l + \pi w = 400$
so $l = \frac{400 - \pi w}{2}$
$= 200 - \frac{\pi}{2} w$

We want to Maximize $A = l \cdot w$

In one variable: $A = \left(200 - \frac{\pi}{2} w\right) \cdot w$

$= 200w - \frac{\pi}{2} w^2$

So, $A' = 200 - \pi w$ & $A' = 0$ if $w = \frac{200}{\pi}$ meters

Note: $A''\left(\frac{200}{\pi}\right) = \frac{200}{\pi}(-\pi) = -200 < 0$, so the only crit pt, $w = \frac{200}{\pi}$ is a max.

Then $l = 200 - \frac{\pi}{2}\left(\frac{200}{\pi}\right)$

$l = 100$ meters

(b) (4 pts) What should the dimensions be if the total area enclosed by the running track is to be as large as possible?

If total area is to be maximized then we want to maximize

$TA = l \cdot w + \pi\left(\frac{w}{2}\right)^2 = lw + \frac{\pi w^2}{4}$

Then $TA = \left(200w - \frac{\pi}{2} w^2\right) + \pi \frac{w^2}{4} = 200w - \frac{\pi w^2}{4}$

& $TA' = 200 - \frac{\pi w}{2}$. Then $TA' = 0$ if $\frac{400}{\pi} = w$ meters

Note: $TA'' = -\frac{\pi}{2} < 0$ for all w, so $w = \frac{400}{\pi}$ is a max. & is the global max. When $w = \frac{400}{\pi}$, $l = 0$ meters, so the track is a circle.

Good luck, and a happy spring and summer to each of you!