(7.) The function \( f(x) = x \ln(x) \) has one critical point on the interval \((0, 5)\).

(a) (4 pts) Determine the exact \( x \) value (i.e., not a decimal approximation) for the location of this critical point.

\[
f'(x) = \ln(x) + x \left( \frac{1}{x} \right) = \ln(x) + 1
\]

\[f'(x) = 0 \text{ if } \ln(x) = -1\]

So \( x = e^{-1} = \frac{1}{e} \)

(b) (3pts) Is this point a maximum or a minimum or neither of these? Explain and show your work.

- **Graphical Argument:**
  - By 2nd deriv. test: \( f''(x) = \frac{1}{x} \) and \( f''(\frac{1}{e}) > 0 \)
  - So \( x = \frac{1}{e} \) is a local minimum.

(c) (2 pts) Determine the instantaneous rate of change of \( f \) at \( x = 1 \) and at \( x = 2 \).

\[
f'(1) = \ln(1) + 1 = 1
\]

\[
f'(2) = \ln(2) + 1 \approx 1.693
\]

\[\text{at } x = 1, \quad f'(1) = 1\]

\[\text{at } x = 2, \quad f'(2) \approx 1.693\]

(d) (2 pts) What do the values in part (c) suggest about the concavity of the function between \( x = 1 \) and \( x = 2 \)? Explain.

- Since \( f' \) increases between \( x = 1 \) and \( x = 2 \), this suggests the function is concave up between \( x = 1 \) and \( x = 2 \).

(e) (3 pts) Determine an equation of the tangent to the graph of \( f \) at \( x = 1 \).

\[\text{at } x = 1, \quad f(1) = 1 \cdot \ln(1) = 0, \quad m = f'(1) = 1,\]

\[y = -0 \cdot (x - 1) \]

\[y = x - 1\]

(f) (2 pts) Use your equation from part (e) to approximate \( f(2) \).

- Using part (e),

\[f(2) \approx 2 - 1 = 1\]

\[\text{[Not}: \frac{\ln(3)}{3^{1/2}} = 1.3716]\]

(g) (2 pts) Should your estimate be an underestimate or an overestimate?

**Underestimate.** Why?

- Because if the function is concave up, the tangent lies below the function.